A convenient method for solving state-space HANK models with sticky expectations (in continuous time)

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Abstract

Full information rational expectations heterogeneous agent models can be easily converted into a sticky expectations environment, even when solved in state-space form. The technique recycles the Jacobians of the full information model with only a few modifications. The process is greatly simplified by working in continuous time, which facilitates the use of natural boundary conditions to ensure agents do not violate idiosyncratic borrowing constraints and the measure of updating agents at any given moment is zero. After solving the full information model, the conversion to sticky expectations takes only a few additional lines of code.

1 Introduction

Deviations from full information rational expectations (often abbreviated as FIRE) are qualitatively and quantitatively important for understanding business cycles and are often necessary to reconcile heterogeneous agent New Keynesian (HANK) models with macroeconomic data. In this supplemental paper to Kwicklis (2025b), I develop a new technique to expediently convert the linearized state-space Jacobians of a full information HANK continuous time system into their sticky expectation counterparts, wherein only a fraction of agents update their beliefs about the macroeconomy to full information at any moment in time. Kwicklis (2025b) then demonstrates an empirical application of the procedure.

Several factors allow my numerical method to offer an expedient solution by recycling FIRE Jacobians. First, the machinery of continuous time naturally handles the interior and boundary of the state-space separately via partial differential equations (PDEs) and their accompanying boundary conditions, which ensure that agents do not violate borrowing constraints and similar restrictions. Second, as explained in Guerreiro (2023), only the average beliefs of the households in the standard sticky information setting matter for aggregate allocations.

As such, my first step is to solve the linearized problem for a household with *average* beliefs about the macroeconomy, given that the average household treats its beliefs (to first order) as the true future when calculating its value function and forming its plan for its control variables. Additionally, in continuous time, only a vanishing measure of households update their beliefs to full information in any given moment, so updates only lead the average belief (and the average behavior it induces) to drift, not jump. Information updates can therefore be incorporated entirely as additive drift terms. Lastly, the average belief value function can be parsimoniously updated using the value functions

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of full information households, as both solve the same partial equilibrium decision problem, just for different (incorrect and correct) sequences of forecasted prices.

I briefly survey related work on HANK models and departures from FIRE in the literature review. In Section 2, I describe the layout of a broad class of sticky expectation HANK models and the mathematical arguments that justify my solution technique. In Section 3, I detail the simple matrix manipulation that implements my methodology. In Section 4, I show that my strategy yields the correct analytical solution for a simple representative agent New Keynesian model that can be solved via pen-and-paper, and that my state-space numerical solution matches the sequence-space approach described in Auclert, Rognlie, and Straub (2020) for a canonical HANK model. Section 5 concludes.

1.1 Literature Review

Several approaches for handling non-FIRE HANK models already exist in the literature, but my methodology offers a flexible and powerful alternative to existing methods. In a seminal paper that merges HANK with sticky expectations, Carroll et al. (2020) use a Krusell and Smith (1998) approach to solve a simple state-space HANK model by tracking the entire distribution of infrequently-updating household expectations. They then demonstrate that their simulated model replicates the empirically observed sluggish response of household consumption to macroeconomic events. However, the authors deliberately keep their model simple due to the computational complexity and rely on specific parametric forms for the utility function and the budget constraint, which my approach does not require. In contrast, Kwicklis (2025b) uses my methodology to solve and estimate more complicated HANK models that are realistic enough for real-time forecasting.

Auclert, Rognlie, and Straub (2020) demonstrate how to conduct a similarly convenient conversion from FIRE to sticky expectations using the sequence-space Jacobian (SSJ) approach of Auclert, Bardóczy, et al. (2021). Like my conversion, their technique also requires only a few small changes to the computation. In Section 3, I provide a numerical example of a canonical HANK model solved with sticky expectations in both my state-space form and in a continuous time variation of their sequence-space methodology. Both approaches generate similar impulse response functions up to a reasonable approximation error. However, while the SSJ framework is a powerful tool, some applications may still be more easily handled in state-space. First, there is a matter of ease of implementation: the SSJ framework involves chaining derivatives across sometimes very complicated directed acyclic graphs. Secondly, determinacy and uniqueness of a stationary solution can be difficult to assess in sequence-space,[3] but are straightforward to assess with the Blanchard and Kahn (1980) methodology in state-space. Thirdly, state-space models sometimes offer advantages for estimation and inference; state-space models can be easily adapted for measurement error, missing data, and changes in the model's governing equations, while a smaller ecosystem of tools is currently available for in sequence-

[3]Most sequence-space determinacy checks involve the use of Onatski (2006) winding criteria and the approximation of the solution with a state-space model. Auclert, Rognlie, and Straub (2023) use the quasi-Toeplitz structure as of the sequence-space Jacobians to approximate their model after a large number of time periods to assess determinacy. Hagedorn (2023) assumes households decisions only depend on aggregate states instead of the full distribution, making the dimension-reduced sequence-space model exactly Toeplitz. However, neither approach directly evaluates the stationarity of the model of interest – only a distant future or dimension reduced approximation.

space. As per Auclert, Bardóczy, et al. (2021), filtering of shock processes is also more theoretically straightforward in state-space than sequence-space. Moreover, state-space models are still widely used by central banks and other institutions (see Acharya et al. (2023) for a HANK example). As such, my methodology provides a recursive state-space alternative to bring non-FIRE expectation structures into HANK settings.

My approach naturally builds off of previous work on solving FIRE state-space HANK models. In my HANK numerical example, I repeatedly employ a continuous time analogue of the approach used in Bayer and Luetticke (2020) to solve for the systems' FIRE Jacobians. Their approach – based on Reiter (2009) – reduces the dimensionality of the heterogeneous agent problem using a discrete cosine transformation for the households' value function and a copula for the distribution of households. Ahn et al. (2018) is also relevant, as it explains perturbation solutions in continuous time HANK models more broadly. Researchers interested in estimating continuous time models from discretely sampled data should further consult Christensen, Neri, and Parra-Alvarez (2024), which provides a guide for properly integrating continuous time equations to discretized measurements of stocks and flows.

In the numerical HANK example, I also draw upon the continuous time tools of Achdou et al. (2021) to calculate the model's non-stochastic steady-state. The model itself is a variation of the one solved in Kwicklis (2025a), but under an active monetary/passive fiscal policy mix, as is conventional in the New Keynesian literature.

2 General Framework

Time is $t \geq 0$ is continuous. Households are ex-post heterogeneous and know the vector of their idiosyncratic state variables $x_t \in \mathcal{X}$ with full information. These state variables are assumed to evolve via a standard stochastic differential equation with the law of motion

$$dx_t = f(x_t, c_t, p_t)dt + \sigma_x(x)dW_t$$

where c_t is the vector of the household's choice of controls, p_t is a vector of macroeconomic variables outside of the individual household's control (like prices or inflation), f is the law of motion governing the state variable's deterministic drift, and $\sigma_x(x)$ is a diagonal matrix through which an independent vector of Brownian motions W_t feeds back into the state equations.[4] Note that f is itself vector-valued; if f_i depends on c, then the coordinate x_i is an endogenous idiosyncratic state variable. If not, then x_i is an exogenous idiosyncratic state variable.

In addition, I assume that idiosyncratic dynamics must satisfy a boundary condition along at least one of its dimensions:

$$x_{j,t} \geq \underline{x}.$$

For simplicity, I assume that $\sigma_{x,j,j}(x) = 0$ if $x_{j,t} = \underline{x}$, such that endogenous idiosyncratic states with a boundary constraint do not evolve with a stochastic diffusion term on the boundary $\partial\mathcal{X}$.

[4]Naturally, the logic in this text can accommodate other kinds of random processes for the state variables, like Poisson jump processes.

Households plan to choose control variables to maximize their expected discounted utility. In contrast to full information rational expectations, however, the household uses its beliefs (indexed by $i \in \mathcal{I}$ with CDF $\Gamma(i)$) about the macroeconomy to forecast the macroeconomy's evolution and its impact on its decision problem, which may or may not be correct. The perceived problem is:

$$\begin{aligned} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = &\max_{(c_\tau^i(x_\tau, p_\tau, \mu_\tau))_{\tau \geq t}} \tilde{\mathbb{E}}_t^i \int_t^\infty e^{-(\tau - t)\rho} u(c_\tau) d\tau \\ \text{s.t.} \quad & \mathbb{E}_t^i[dx_t | dW_t] = \mathbb{E}_t^i[f(x_t, c_t; p_t) dt] + \sigma_x(x) dW_t, \\ & \mathbb{E}_t^i[\partial_t \mu_t] = \mathbb{E}_t^i[\mathcal{D}_t^*(V, p)[\mu_t](x)] \\ & \frac{\mathbb{E}_t^i[dp_t]}{dt} = \mathbb{E}_t^i[g(\mu_t, p_t)] \\ & x_{jt} \geq \underline{x} \; \forall t \geq 0 \end{aligned} \tag{1}$$

where $\mathcal{D}^*(V, p)$ is the true infinitessimal generator for the Kolmogorov Forward equation (KFE) of the distribution μ, while g is the true law of motion for aggregates p. $\tilde{\mathbb{E}}^i$ is the expectation taken with the subjective probability measure of a household with belief i at time t. Here, $p_t^i \equiv \tilde{\mathbb{E}}_t^i[p_t]$ and $\mu_t^i \equiv \tilde{\mathbb{E}}_t^i[\mu_t]$

Although households may have incorrect beliefs about the trajectory of prices, they only use those incorrect beliefs for forecasting and constructing their value function V_t^i. For the first-order conditions that arise from their decision problem in the interior $\mathcal{X} \setminus \partial\mathcal{X}$, I assume households plug the actual p_t into their choices at time t, such that

$$c_t^i(x) = h(x, V_t^i, p_t).$$

These consumption choices and contemporaneous prices are assumed to have no impact on the household's value function forecast. Similarly, I assume that households on a boundary $\partial\mathcal{X}$ with $x_{j,t} = \underline{x}$ choose consumption according to

$$f_j(x_t, h(V_t^i, p_t), p_t) = 0.$$

This is tantamount to a sequence of boundary constraints for the value function over time. In later sections, I show that this boundary condition is implied by a "mass-preserving" KFE infinitessimal generator and does not need to be imposed directly.

As in the sticky information framework of Caroll et al (2020) and Auclert et al (2023) (MJMH), households in each moment either update to full information about the aggregate system or not at all for the purpose of constructing V_t^i. They do so with a constant, independent Poisson intensity λ; in an infinitessimal increment of time, a random λdt mass from the cross section of households updates to full information.

While households are able to reason through the dynamics of g given their beliefs about its inputs, I assume that the p equations follow the general structural relation

$$Q dp_t = q(\mu_t, p_t, \{h(V_t^i, p_t)\}_{i \in \mathcal{I}}) dt. \tag{2}$$

If Q is invertible, then $g = Q^{-1} q$. In other cases some rows of Q are entirely zero, such that the equation denotes a static fixed point relationship for which g is a solution. In this way, equation (2) encompasses the dynamics of macroeconomic jump variables (like inflation), macroeconomic state variables (like the capital stock), and static variables (like prices).

2.1 Households in the Interior

I consider the interior $\mathcal{X} \setminus \partial\mathcal{X}$ and $\partial\mathcal{X}$ separately, as in the former case the i-indexed beliefs affect households' decisions, while in the latter they do not. Working with the two cases separately is straightforward in continuous time, as the recursive Hamilton Jacobi Bellman (HJB) equation describes only the state-space's interior.

Discretizing the value function (1) in with infinitessimal time increments dt, the analogue to the discrete time value function is

$$V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{(c_\tau^i)_{\tau \geq t}} u(c_t^i)dt + e^{-\rho dt}\tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}, p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) +$$
$$\text{s.t. } \mathbb{E}_t^i[dx_t|dW_t] = \mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \sigma_x(x)dW_t,\ \dot{p}_t^i = \mathbb{E}_t^i[g(\mu_t, p_t)],\ \partial_t\mu_t^i = \mathbb{E}_t^i[\mathcal{D}_t^*(V, p)\mu_t]$$

where conditioning on the subjective p^i, μ^i, the evolution of p, μ is irrelevant for the household's decision problem (although the *level* is still relevant).

Proposition 2.1. *The Hamilton Jacobi Bellman (HJB) equation for $x \in \mathcal{X} \setminus \partial\mathcal{X}$ takes the form*

$$\rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{\tilde{c}_t^i} \Big\{ u(\tilde{c}_t^i) + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \frac{1}{2} tr(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt$$
$$+ \nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\tilde{\mathbb{E}}_t^i[g(p, \mu)] + \int_{\mathcal{X}} \delta_{\mu(x')} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i[\mathcal{D}_t^*(V, p_t)\mu_t(x')]dx' \Big\}. \tag{3}$$

Here, I write $\delta_{\mu(x')}F(x)$ as a shorthand for $\frac{\delta F(x)}{\delta \mu(x')}$, the functional (Frechét) derivative of $F(x)$ with respect to $\mu(x')$.

Proof. See Appendix A.1. □

Definition 2.2. I define a *non-stochastic steady state* as one in which the value function no longer explicitly depends on time, and the macroeconomic variables μ, p are equal to their expected values across the entire economy and are no longer changing. All households have the correct belief, while $\dot{p} = 0$ and $\partial_t\mu_t = 0$.

Up to a first-order approximation in the macroeconomic variables around the non-stochastic steady state, the household will treat its forecast for prices and the distribution as if they are the true prices and distribution, as in Carroll et al. (2020). As such, I can write the HJB as a function of p_t^i and μ_t^i alone:

$$\rho V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i) = \max_{\tilde{c}_t^i} \Big\{ u(c_t^i) + \nabla_x V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i)' f(x_t, c_t^i; \tilde{p}_t^i)$$
$$+ \nabla_{p^i} V_t^i(x_t, \tilde{p}_t^i, \tilde{\mu}_t^i)' g(\tilde{p}_t^i, \tilde{\mu}_t^i) \tag{4}$$
$$+ \int_{\mathcal{X}} \delta_{\mu^i(x')} V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i(x'))\mathcal{D}_t^*(V_t^i(x'), \tilde{p}_t^i)\tilde{\mu}_t^i(x')]dx' \Big\}$$

Note that the dependence of the value function on i is entirely through the expected aggregates p^i and μ^i, while actual p and μ do not affect the problem. The interior household will plan to choose consumption to maximize its expected utility only using its subjective beliefs about prices, the distribution, and other macro aggregates. Assuming that the optimization problem is concave, the household's planned control choice will thereby satisfy

$$\nabla u(\tilde{c}_t^i) = -\nabla_x V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i)' \partial_{c^i} f(x_t, \tilde{c}_t^i; \tilde{p}_t^i).$$

For many problems, the budget constraint can be rewritten so that the household chooses only a numeraire consumption good, as in Caroll et al (2020). In such cases, c can be written entirely in terms of $\nabla_x V^i$. In more complicated settings, however, one could consider cases with variable control prices that the consumer is able to observe (but does *not* use to update their forecast). The consumer *actually* choose c^i such that

$$\nabla u(c_t^i) = -\nabla_x V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i)' \partial_{c^i} f(x_t, c_t^i; p_t),$$

which equates the instantaneous value of the control with its perceived opportunity cost (e.g. the value of consumption with the *subjective* value of the savings given actual present prices). If p_t does *not* change the consumption plan, however, it still does not enter into the forecast of V_t^i. Rather, actual p_t only enters into how the distribution is updated.

I denote this control variable choice that satisfies the FOC

$$c_t^i(x_t; p_t, \mu_t, \tilde{p}_t^i, \tilde{\mu}_t^i) = h\Big(V_t^i(x_t; \tilde{p}_t^i, \tilde{\mu}_t^i), p_t\Big).$$

2.2 The Average-Belief Household

At this stage, it is useful to define an agent with *average* beliefs about the state of the macroeconomy. This agent does not actually exist; in the model, households either have full information or don't following a macroeconomic shock. Still, the construct is useful, as the average agent will behave as if the average beliefs about prices are the true dynamics, and their value function can be used to determine the average choices in the economy at every point in the idiosyncratic state-space $\mathcal{X}$. To see why this is useful and convenient, I show that in a first-order expansion, only the average belief matters for the households' aggregate control variables, as Guerreiro (2023) argues in a sequence-space setting.

From there, I derive the evolution of average beliefs under sticky expectations in continuous time. The result is a system of intuitive and tractable differential equations that are straightforward to add to the model.

2.2.1 The Average-Belief Value Function

Recall that the economy is populated with agents who have beliefs indexed by i about macroeconomic states like the distribution μ and prices and aggregates p, and denote this subjective probability density $\tilde{\psi}^i_t$. Subjective expectations about a macroeconomic random variable Y are calculated with the subjective measure:

$$\tilde{\mathbb{E}}^i_t[Y] \equiv \int_S y\tilde{\psi}^i_t(y)dy$$

Let the measure of housholds with belief i be $\Gamma(i)$. Define the average belief about a macroeconomic variable Y as

$$\overline{\mathbb{E}}_t[Y] \equiv \int_S y\overline{\psi}_t(y)dy$$

where $\overline{\psi}_t(y) \equiv \int_i \tilde{\psi}_i(y)d\Gamma(i)$ is the average agent's belief about the PDF of Y at time t. Just like $\tilde{p}^i$ and $\tilde{\mu}^i$ were calculated using the i probability measures, I similarly define $\overline{p}_t$ and $\overline{\mu}_t$ as the *average* beliefs about the macroeconomy:

$$\overline{p}_t \equiv \overline{\mathbb{E}}_t[p_t],\ \overline{\mu}_t \equiv \overline{\mathbb{E}}_t[\mu_t].$$

Define the "expectations-averaged" value function over the entire population *before* any agents update their beliefs as

$$\begin{aligned}\rho\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t) = &\max_{\overline{c}_t}\Big\{u(\overline{c}_t) + \nabla_x\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t)'f(x_t,\overline{c};\overline{p}_t)\\ &+ \nabla_{\overline{p}}\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t)'g(\overline{p}_t,\tilde{\mu}_t) + \int_{\mathcal{X}}\delta_{\overline{\mu}(x')}\overline{V}_t(x_t;\overline{p}_t,\overline{\mu}_t)\mathcal{D}^*_t(\overline{V}_t,\overline{p}_t)[\overline{\mu}_t](x')dx'\Big\} \\ &\text{s.t. } x_{j,t} \geq \underline{x}\ \forall t\end{aligned} \tag{5}$$

The value function is simply the preceding agents' value function, but specifically using the *average* belief about macroeconomic variables.

Proposition 2.3. *As in Guerreiro (2023), the value function averaged over beliefs $\overline{V}_t$ and actual prices and aggregates p_t characterize the average control variable choice given the households' idiosyncratic states to first order. In other words, in a neighborhood around the non-stochastic steady-state with deviations thereof denoted by Δ,*

$$h(\overline{V}_t,p_t) = \int_i c^i_t(x;p,\mu,p^i,\mu^i)d\Gamma(i) + \mathcal{O}(\|\Delta p^2,\Delta\mu^2,(\Delta p^i)^2,(\Delta\mu^i)^2\|).$$

Proof. See Appendix A.3. □

2.2.2 Sticky Expectations in Continuous Time

For a random variable that is changing over time, the total change in the average forecast over time will be

$$\begin{aligned}\frac{d}{dt}\left(\overline{\mathbb{E}}_t[Y_{t+s}]\right) &= \frac{d}{dt}\left(\int_\Omega y_{t+s}(\omega)\overline{\psi}_t(\omega)d\omega\right)\\ &= \underbrace{\int_\Omega \dot{y}_{t+s}(\omega)\overline{\psi}_t(\omega)d\omega}_{\text{Subj. Forecast}} + \underbrace{\int_\Omega y_{t+s}(\omega)\partial_t\overline{\psi}_t(\omega)dy_t}_{\text{"Gain"}}\\ &= \overline{\mathbb{E}}_t[\dot{Y}_{t+s}] + \frac{d\overline{\mathbb{E}}}{dt}[Y_{t+s}]\end{aligned}$$

This structure has a Kalman Filter-like intuition: the agents' ex-post belief about a macroeconomic variable For a sticky-information environment like the one detailed in Mankiw and Reis (2002), Carroll et al. (2020), Auclert, Rognlie, and Straub (2020), and many others, the average belief is:

$$\overline{\mathbb{E}}_t[Y_{t+s}] = \int_0^\infty \lambda e^{-\lambda\tau}\mathbb{E}_{t-\tau}[Y_{t+s}]d\tau.$$

Differentiating with respect to t, I show in the Appendix A.2 that the average expectation then follows

$$\frac{d}{dt}\overline{\mathbb{E}}_t[Y_{t+s}] = \overline{\mathbb{E}}_t[\partial_t Y_{t+s}] + \lambda\Big(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]\Big).$$

Crucially, when a household updates from stale beliefs about the macroeconomy to full information, they do not just update their forecast for the variable at time t. Rather, they update their entire sequence of forecasts for the entire future, such that the update takes the form of an entire sequence of revisions
$\big\{\lambda(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]),\ s \geq 0\big\}$.

While the change in the entire forecast sequence is crucial for proper updating, I later show that it suffices to track just the zero-horizon forecasts for macroeconomic variables. For prices and aggregates p_t in the economy and the distribution μ_t, I define the zero-horizon expected values

$$\overline{p}_t \equiv \lim_{dt\to 0} \overline{\mathbb{E}}_t[p_{t+dt}]$$

$$\overline{\mu}_t \equiv \lim_{dt\to 0} \overline{\mathbb{E}}_t[\mu_{t+dt}].$$

The zero-horizon forecasts will then evolve according to

$$\frac{d\overline{p}_t}{dt} = \overline{\mathbb{E}}_t[\partial_t p_t] + \lambda(p_t - \overline{p}_t)$$

$$\frac{d\overline{\mu}_t}{dt} = \overline{\mathbb{E}}_t[\partial_t \mu_t] + \lambda(\mu_t - \overline{\mu}_t)$$

where the first term of each expression is the agent's perceived belief of how prices and the distribution evolve before new information updates agents' forecasts. In other words, to first order the average belief is updated as follows:

$$\frac{d\overline{p}_t}{dt} = g(\overline{p}_t, \overline{\mu}_t, \overline{V}) + \lambda(p_t - \overline{p}_t) \tag{6}$$

$$\partial_t\overline{\mu}_t(x) = \mathcal{D}_t^*(\overline{V}, \overline{p})[\overline{\mu}_t](x) + \lambda(\mu_t(x) - \overline{\mu}_t(x)) \tag{7}$$

2.3 The Distribution of Agents (and Households on the Boundary)

Up to this point, I have referenced the $\mathcal{D}^*$ infinitessimal generator; I now define it explicitly and discuss how it implicitly enforces the boundary constraints referenced in the Overview section. Consider the average value function $\overline{V}$ that induces consumption choices according to the first-order conditions $c_t(x; p_t, \overline{p}_t, \overline{\mu}_t) = h(\overline{V}_t(x; \overline{p}_t, \overline{\mu}_t), p_t)$. For an agent on the boundary $\partial\mathcal{X}$, e.g. at a borrowing constraint, the appropriate boundary condition on $\overline{V}$ to describe the agent's behavior is

$$f_i(x, h(\overline{V}_t, p_t); p_t, \mu_t) = 0 \quad \text{if } x_i = \underline{x} \tag{7.5}$$

such that $V_t(x)$, $x \in \partial\mathcal{X}$ satisfies the above implicit relationship. (Technically, this constraint should hold for every V_t^i, but only $\overline{V}$ is computationally important). In the particular context of a borrowing constraint, the above implies that the household consumes exactly its income when its assets are zero – such that the asset state variable does not drift past the constraint.

To enforce the appropriate sequence of boundary conditions on the HJB relationship, one need only ensure that the distribution μ whose mass starts within $\mathcal{X}$ stays within $\mathcal{X}$.

First, note that the evolution of the distribution of households with the value function V may be expressed via a standard Kolmogorov Forward Equation (KFE)

$$\partial_t \mu_t(x) = -\nabla_x \cdot (f(x, h(\overline{V}_t, p_t); p_t)\mu_t(x)) + \nabla^2 \text{tr}\left[(\sigma(x)\sigma(x)'\mu_t(x))\right] \tag{8}$$

given the $\sigma(x)$ diffusion matrix is diagonal. Equation (27) depends on the *actual* prices and the *actual* distribution. Expectations about prices only enter into the households' value function V, which may reflect some more complicated information or belief structure. This equation can be more compactly represented with the KFE infinitessimal generator operator $\mathcal{D}^*$, such that

$$\partial_t \mu_t = \mathcal{D}^*(\overline{V}_t, p_t)[\mu](x).$$

Definition 2.4. Define the KFE operator's *kernel* $D^*(V, p)(x, y) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that

$$\mathcal{D}^*(V, p)[\mu](x) = \int_{\mathcal{X}} D^*(V, p)(x, x')\mu(x')dx'.$$

Definition 2.5. A KFE infinitessimal generator $\mathcal{D}^* : F[\mathcal{X}] \to F[\mathcal{X}]$ is *mass-preserving* if its kernel satisfies

$$\int_{\mathcal{X}} D^*(V, p)(x, x')dx = 0 \quad \forall x' \in \mathcal{X}.$$

By analogy, let $d^* = [d^*_{i,j}]$ be a matrix finite difference approximation the kernel of $\mathcal{D}^*(V, p)$, e.g. $D^*(V, p)(x, y)$. d^* will be mass-preserving if all of its columns sum to zero, such that:

$$\sum_i d^*_{i,j} = 0.$$

Proposition 2.6. *Suppose the economy starts in its non-stochastic steady state when a macroeconomic shock occurs. If the KFE generator is mass-preserving, then the value function of households at the boundary will satisfy equation (7.5).*

Proof. See Appendix A.4. □

The proof goes roughly as follows: if the KFE operator is mass-preserving, then the net flux across the boundary defined by the state constraints must be zero for all time. If all the probability mass starts within or on the boundary of the space, then no mass crosses the boundary, and so probability mass exactly on the boundary must be traveling tangent to it. This tangent motion is equivalent to the value function satisfying equation (7.5).

Tracking the evolution of the distribution with a mass preserving KFE operator therefore naturally imposes a time-varying boundary condition for the value function that depends on the realization of actual aggregates. For example, if households are unable to borrow and are at a constraint of 0 assets, they will consume a maximum of their current income, regardless of their beliefs.

For most applications involving a first-order perturbation solution, the mass-preserving KFE generator will indeed be mass-preserving.

Proposition 2.7. *If the KFE infinitessimal generator $\mathcal{D}^*(V,p)$ is a first-order perturbation of the steady-state one with respect to macroeconomic variables, then it will be mass-preserving if the Jacobians evaluated at the steady-state are mass-preserving.*

Proof. See Appendix A.5. □

The result follows immediately from the linearity of the operators. In effect, if the perturbation solution enforces the correct idiosyncratic constraints in the FIRE case via the KFE operator, it will also enforce the correct constraints in the non-FIRE case along the boundary.

2.4 Aggregation

Because the average belief value function determines the value of average controls conditional on the idiosyncratic state-space point in the state-space, $\overline{V}$ will also be sufficient to characterize macroeconomic aggregates. Aggregate controls will then be

$$C_t = \int_{\mathcal{X}} h(\overline{V}_t, p_t)\mu_t(x)dx$$

such that the aggregate variables will depend on the *actual measure* of individuals given their choices derived from the average expectations. Similarly, aggregate states can be computed as

$$X_t = \int_{\mathcal{X}} x\mu_t(x)dx.$$

As such, the actual p_t in the economy will evolve with the true distribution μ_t, the true p_t, and the subjective belief-averaged $\overline{V}_t$:

$$Qdp_t = q(\mu_t, p_t, h(\overline{V}_t, p_t)dt. \tag{9}$$

2.5 Full Information Households

Every period, a mass λdt mass od new households becomes full information. It's necessary to track these households as well, as they become a greater and greater share of the population over time (and thus have a greater and greater effect on the average). Define the full information value function as

$$
\begin{aligned}
\rho\widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t) = \max_{c_t}\Bigg\{ & u(c_t) + \nabla_x\widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)' f(x_t,c_t;p_t) \\
& + \nabla_p\widehat{V}(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)'\frac{\mathbb{E}[dp_t]}{dt} + \int_{\mathcal{X}}\delta_\mu\widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)\partial_t\mu_t dx \\
& + \nabla_{\overline{p}}\widehat{V}(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)'\frac{d\overline{p}_t}{dt} + \int_{\mathcal{X}}\delta_{\overline{\mu}}\widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{\mu}_t,\overline{V}_t)\partial_t\overline{\mu}_t dx \\
& + \int_{\mathcal{X}}\delta_{\overline{V}}\widehat{V}_t(x_t;p_t,\mu_t,\overline{p}_t,\overline{V}_t)\partial_t\overline{V}_t dx\Bigg\}.
\end{aligned} \tag{10}
$$

$\widehat{V}$ uses the true law of motion for prices – given the actual prices and the mean beliefs across the economy, and the belief-averaged value function $\overline{V}$, which influences average choices. A rational full information household thus forecasts 1) their idiosyncratic state variables' evolution, given the true prices, 2) the evolution of those prices, given the true distribution of agents and their average choices (encapsulated by average belief $\overline{V}$), 3) the evolution of the total distribution, 4) the evolution of expected prices and 5) expected distributions for the average household, and 6) the average value function (and therefore decisions) of the average agent in the economy, which when combined with the true distribution is used to formulate a forecast for prices.

Altogether, equations (5-10) nearly describe how the system evolves for the purposes of calculating macroeconomic and microeconomic (but expectations-averaged) variables, but with a caveat: equation (5) is incomplete, and only models the average household dynamics if the composition of households did not change. With a probability λdt, a household is uniformly selected (after choosing their consumption) to update their beliefs about the macroeconomic variables to full information. As such, $\overline{V}$ should evolve according equation (5) – but with an additional $\lambda(\widehat{V}_t - \overline{V})$ that the average household does not anticipate or plan for in their optimization problem. I discuss how to incorporate this adjustment into the dynamics in the next section.

2.6 A two-part problem

To solve the model, one can solve for two different stable manifolds (subspaces in the linearized model), consecutively. First, one can solve for the behavior of the fictitious average agent to determine the evolution of $\overline{V}$. Then, one can solve the full information households' problem, taking the average agent as a state variable (and where the full information agents internalize how they will update the average agent over time).

2.6.1 The mean belief household solution

First, consider the perceived problem of the fictitious average household. For a given household, the value function may be concentrated to have an explicit time dependence, such that it represents the choices of the household for a given sequence of macroeconomic aggregates. Denoting the drift of the macroecomic variables $\partial_t \overline{V}(x_t)$, one can write

$$\partial_t \overline{V}(x_t) = \nabla_{\overline{p}} \overline{V}_t(x_t; \overline{p}_t, \overline{\mu}_t)' g(\overline{p}_t, \tilde{\mu}_t) + \int_{\mathcal{X}} \delta_{\overline{\mu}} \overline{V}_t(x_t; p_t, \mu_t, \overline{p}_t, \overline{\mu}_t) \mathcal{D}_t^*(\overline{V}_t, \overline{p}_t)[\overline{\mu}_t] dx.$$

By subsuming the macroeconomic variable dependence into the value function, the decision problem or "partial equilibrium" value function is then

$$\begin{aligned} \rho \overline{V}_t(x_t) = &\max_{\overline{c}_t} \left\{ u(\overline{c}_t) + \nabla_x \overline{V}_t(x_t)' f(x_t, \overline{c}; \overline{p}_t) \right\} + \partial_t \overline{V}(x_t) \\ &\text{s.t. } x_t \geq \underline{x} \ \forall t \end{aligned} \tag{11}$$

where

$$\partial_t \overline{\mu}_t = \mathcal{D}_t^*(\overline{V}_t, \overline{p}_t) \mu_t, \tag{12}$$

$$Q \frac{d\overline{p}_t}{dt} = q(\overline{\mu}_t, \overline{p}_t, \overline{V}_t). \tag{13}$$

This system exactly resembles the FIRE system – except with expected prices in lieu of the real ones. The reason for this is that the average expectation agent believes that their forecast of prices is correct (or at least, on average correct in a certainty equivalent setting).

The concentrated HJB can then be linearized around the non-stochastic steady-state with respect to the macroeconomic variables as

$$\partial_t \Delta \overline{V}(x_t) = \int_{\mathcal{X}} \mathcal{A}_{VV}(x, x') \Delta \overline{V}_t(x') dx' + \int_{\mathcal{X}} \mathcal{A}_{V\mu}(x, x') \Delta \overline{\mu}_t(x') dx' + \mathcal{A}_{Vp}(x) \Delta \overline{p}_t + \mathcal{O}([...]^2). \tag{14}$$

The $\mathcal{A}$ operators denote the partial equilibrium Jacobians of the household's concentrated HJB with respect to its own value function and the average beliefs about prices and the distribution evaluated in the non-stochastic steady state. In other words,

$$\mathcal{A}_{VV}(x, x') = \rho \delta(x - x') - \frac{\delta}{\delta v(x')} \Big[h(v(x)) + \nabla_x v(x)' f(x, h(v(x)); \overline{p}) \Big],$$

$$\mathcal{A}_{V\mu}(x,x') = -\frac{\delta}{\delta\overline{\mu}(x')}\Big[h(v(x)) + \nabla_x v(x)' f(x, h(\overline{V}); p)\Big] \ (= 0).$$

$$\mathcal{A}_{Vp}(x) = -\nabla_x v(x)' \frac{\partial}{\partial p} f(x, h(v); p).$$

where $\delta(x - x')$ is a Dirac-delta function and $\frac{\delta f(x,g(x))}{\delta g(x')}$ refers to the functional (Frechét) derivative of f with respect to g. Note that in the steady-state, actual and expected prices are equal and all households have the same value function $v(x)$; these Jacobians are exactly the same as their FIRE counterparts.

Suppose the average household's perceived problem can be solved for the value function's dynamics on the stable manifold, such that for a sequence of beliefs about prices and the distribution, the value function will satisfy (at least, under the household's average beliefs)

$$\overline{\mathbb{E}}_t[\partial_t \Delta\overline{V}(x_t)] = \int_{\mathcal{X}} \mathcal{B}_{VV}(x,x')\Delta\overline{V}_t(x_t')dx' + \int_{\mathcal{X}} \mathcal{B}_{V\mu}(x,x')\Delta\overline{\mu}_t(x')dx' + \mathcal{B}_{Vp}(x)\Delta\overline{p}_t + \mathcal{O}([...]^2).$$

The forecasts of the expected household (prior to updating) will also be

$$\overline{\mathbb{E}}_t[\partial_t \Delta\mu_t(x)] = \int_{\mathcal{X}} \mathcal{B}_{V\mu}(x')\Delta\overline{V}_t(x_t')dx' + \int_{\mathcal{X}} \mathcal{B}_{\mu\mu}(x,x')\Delta\overline{\mu}_t(x')dx' + \mathcal{B}_{\mu p}(x)\Delta\overline{p}_t + \mathcal{O}([...]^2)$$

$$\overline{\mathbb{E}}_t[\partial_t \Delta p_t] = \int_{\mathcal{X}} \mathcal{B}_{pV}(x,x')\Delta\overline{V}_t(x_t')dx' + \int_{\mathcal{X}} \mathcal{B}_{p\mu}(x')\Delta\overline{\mu}_t(x')dx' + \mathcal{B}_{pp}\Delta\overline{p}_t + \mathcal{O}([...]^2)$$

In the actual economy, however, the average beliefs are updated with the realizations of the actual p_t and μ_t. Unfortunately, this is slightly complicated by the fact that learning at time t updates the whole forecast sequence of $(\overline{p}_\tau, \overline{\mu}_\tau)_{\tau \geq t}$, not just their contemporaneous values. To see why this is important, consider integrating forward equation (14), with the assumption that $\lim_{t\to\infty} \Delta V_t(x) = 0$. Using the partial equilibrium Jacobians and treating the linear operators analogously to matrices, the value function becomes:

$$\Delta\overline{V}_t(x) = \int_t^\infty \int_{\mathcal{X}} \big[e^{-\mathcal{A}_{VV}(\tau - t)}\big](x,x'')\Big[\int_{\mathcal{X}} \mathcal{A}_{V\mu}(x,x')\Delta\overline{\mu}_\tau(x')dx' + \mathcal{A}_{Vp}\Delta\overline{p}_\tau\Big]dx''d\tau.$$

where the exponential operator is $\big[e^{-\mathcal{A}_{VV}t}\big](x,x')$ is the kernel equivalent to a matrix exponential.[5] An update results in a change to the value function that takes the entire future path of the new forecast into the account – a complicated object. Fortunately, there's a simpler approach: use the present values already calculated in the value functions of the full information agents.

2.6.2 Full information households and updating

Consider the full information agent's decision problem, given a sequence of macro aggregates and the behavior and beliefs of other agents in the economy (e.g. $\overline{V}, \overline{\mu}, \overline{p}$). By concentrating equation (10),

[5] More explicitly, $\big[e^{\mathcal{A}_{VV}t}\big](x,x') \equiv \delta(x - x') + \sum_{n=1}^{\infty} \frac{t^n}{n!}\mathcal{A}_{VV}^{(n)}(x,x')$,
where $\mathcal{A}_{VV}^{(n)}(x,x') \equiv \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \mathcal{A}_{VV}(x,x_1)\mathcal{A}_{VV}(x_1,x_2)(\ldots)\mathcal{A}_{\mathcal{V}\mathcal{V}}(x_{n-1},x')dx_1 \ldots dx_{n-1}$.

the full information value function is

$$\rho \widehat{V}_t(x_t) = \max_{c_t} \left\{ u(c_t) + \nabla_x \widehat{V}_t(x_t)' f(x_t, c_t; p_t) \right\} + \partial_t \widehat{V}(x_t)$$
$$\text{s.t. } x_t \geq \underline{x} \ \forall t. \tag{15}$$

Now, however, the sequence of actual prices evolve using the *average* value function (which characterizes average household actions) and the actual distribution, along with actual prices and the actual distribution.

$$\partial_t \mu_t = \mathcal{D}^*(\overline{V}_t, p_t)\mu_t$$

$$Q\dot{p}_t = q(\mu_t, p_t, \overline{V})$$

The full information rational agent knows that the other agents will learn over time. The law of motion for the average macroeconomic beliefs is the solution to the non-updating household's problem, but modified for the λdt measure of agents that update using the true values. As such,

$$\frac{d\Delta \overline{p}_t}{dt} = \overline{\mathbb{E}}_t[\Delta \partial_t p_t] + \lambda(\Delta p_t - \Delta \overline{p}_t) \tag{16}$$

$$\frac{\partial \Delta \overline{\mu}_t}{\partial t} = \overline{\mathbb{E}}_t[\partial_t \Delta \mu_t] + \lambda(\Delta \mu_t - \Delta \overline{\mu}_t) \tag{17}$$

where $\overline{\mathbb{E}}_t[\partial_t \Delta p_t]$ and $\overline{\mathbb{E}}_t[\partial_t \Delta \mu_t]$ are the solutions from the average expectation block. By analogy, one could reasonably guess:

$$\frac{\partial \Delta \overline{V}_t}{\partial t} = \overline{\mathbb{E}}_t[\partial_t \Delta V_t] + \lambda(\Delta \widehat{V}_t - \Delta \overline{V}_t)$$

where $\overline{\mathbb{E}}_t[\partial_t V_t]$ is again the solution from the average belief households' problem. This turns out to be correct, as per the following proposition:

Proposition 2.8. *To a first-order approximation, the average belief household updates its value function with a constant factor of* $\lambda(\Delta \widehat{V} - \Delta V)$.

Proof. See Appendix A.6. □

The intuition behind the result is straightforward: although rational expectations households and non-updating households have very different information sets, both solve essentially the same partial equilibrium decision problem when planning their consumption, just with different beliefs. The value functions themselves are linearized with respect to those beliefs, so the effect of a change in a sequence of beliefs is equal to a difference between value functions.

One could also think about the intuition in a slightly different, but equivalent, way: as time progresses following a time-zero shock, the mass of households who have updated grows at a rate of λ per unit of time, while the mass who think they are still in the steady state shrinks at the same rate. This pulls the overall average belief households toward the full information ones at a rate of λ, as more and more FIRE households become averaged into the entire population.

3 Linearized Solution

The preceding section described the linearized solution in a more abstract functional form. To actually calculate the solution on the computer, one discretizes the functions onto grids as described in Achdou et al (2020). Functions become vectors, while integrals becomes sums.

Altogether, the process can be summarized in three steps:

1. Solve the full information rational expectations model:

$$\begin{aligned} \rho V_t(x_t) = &\max_{c_t} \left\{ u(c_t) + \nabla_x V_t(x_t)' f(x_t, c_t; p_t) \right\} + \partial_t V_t(x_t) \\ &\text{s.t. } x_t \geq \underline{x} \; \forall t \\ &\partial_t \mu_t = \mathcal{D}_t^*(p_t, V_t)\mu_t \\ &Q\dot{p}_t = q(\mu_t, p_t, V_t) \end{aligned} \tag{18}$$

2. Construct the solution to the average belief households' problem in the absence of updating. This is simply the FIRE solution, but with the subjectively expected variables instead of the true ones. The new system describes $\overline{\mathbb{E}}_t[\partial_t V_t]$, $\overline{\mathbb{E}}_t[\partial_t \mu_t]$, and $\overline{\mathbb{E}}_t[\partial_t p_t]$.

3. Solve the full information households' rational expectations problem given the average behavior of the other agents, accounting for how the average information agent updates.

In what follows, I assume knowledge Bayer and Luetticke (2020) and Ahn et al. (2018), which are in turn based on the methodology of Reiter (2009). After discretizing the value functions and distributions over a grid, one may solve for the non-stochastic steady-state. Thereafter, one constructs a first-order perturbation of the economy from that steady-state due to aggregate shocks. The A and B block matrices are essentially the discretized matrix representations of the $\mathcal{A}$ and $\mathcal{B}$ terms introduced earlier in the text. I also dispense with the Δ notation; V, μ, and p in this section are all discretized vectors that represent deviations from the non-stochastic steady-state.

First, one starts with the FIRE Jacobians for equation (18):

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} \mathbb{E}[dV] \\ d\mu \\ \mathbb{E}[dp] \end{bmatrix} = \begin{bmatrix} A_{VV} & A_{V\mu} & A_{Vp} \\ A_{\mu V} & A_{\mu\mu} & A_{\mu p} \\ A_{pV} & A_{p\mu} & A_{pp} \end{bmatrix} \begin{bmatrix} V \\ \mu \\ p \end{bmatrix} dt, \tag{19}$$

If the Blanchard and Kahn (1980) conditions are satisfied, one can solve the system as in Sims (2002) using a generalized Schur decomposition to determine its dynamics on its stable manifold – the stable subspace in the linearized, discretized model. The solved rational expectations model is then

$$\begin{bmatrix} dV \\ d\mu \\ dp \end{bmatrix} = \begin{bmatrix} B_{VV} & B_{V\mu} & B_{Vp} \\ B_{\mu V} & B_{\mu\mu} & B_{\mu p} \\ B_{pV} & B_{p\mu} & B_{pp} \end{bmatrix} \begin{bmatrix} V \\ \mu \\ p \end{bmatrix} dt. \tag{20}$$

Once again, the B_{ij} matrices represent the Jacobians of the equilibrium system, restricted to the stable subspace.

Before any updating occurs, agents behave with the belief that the feedbacks of the system are in the stable subspace spanned by the B system in the absence of shocks. Over time, however, households are awakened with a Calvo Poisson rate of λ to the fact that a shock has perturbed the economy from its non-stochastic steady-state. The linearized average beliefs about prices and the distribution then evolve according to

$$\frac{d\overline{\mu}_t}{dt} = B_{\mu V}V_t + B_{\mu V}\overline{\mu}_t + B_{\mu p}\overline{p}_t + \lambda(\mu_t - \overline{\mu}_t)$$

$$\frac{d\overline{p}_t}{dt} = B_{pV}V_t + B_{pV}\overline{\mu}_t + B_{pp}\overline{p}_t + \lambda(p_t - \overline{p}_t).$$

$$\frac{d\overline{V}_t}{dt} = B_{VV}\overline{V}_t + B_{V\mu}\overline{\mu}_t + B_{Vp}\overline{p}_t + \lambda(\widehat{V}_t - \overline{V}_t).$$

with the initial conditions $\overline{p}_0 = 0$, $\overline{\mu}_0 = 0$, and $\overline{V}_0 = 0$ if the agents start with the belief that no shocks have occured such that they are in the non-stochastic steady-state. As discussed in the preceding sections, the distribution evolves according to the average control choices induced by the average belief. These affect the prices in the economy, which are determined via linearized market clearing conditions. Given the actual distribution (and the actual value of other macroeconomic variables), prices thus solve the same fixed point problem that they do in rational expectations – except that now, they must be consistent with market clearing under the evolution of control variables chosen with the non-FIRE belief-averaged value function:

$$A_{pp}p_t + A_{p\mu}\mu_t + A_{pV}\overline{V}_t = Qdp/dt$$

Actual prices in turn determine the actual decision problem for the full information value function $\widehat{V}$, which is sufficient for updating the average belief value function $\overline{V}$.

Altogether, the new system for the sticky expectation economy is:

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbb{E}[d\widehat{V}] \\ d\mu \\ \mathbb{E}[dp] \\ d\overline{V} \\ d\overline{\mu} \\ d\overline{p} \end{bmatrix} = \begin{bmatrix} A_{VV} & A_{V\mu} & A_{Vp} & 0 & 0 & 0 \\ 0 & A_{\mu\mu} & A_{\mu p} & A_{\mu V} & 0 & 0 \\ 0 & A_{p\mu} & A_{pp} & A_{pV} & 0 & 0 \\ \lambda I & 0 & 0 & B_{VV} - \lambda I & B_{V\mu} & B_{Vp} \\ 0 & \lambda I & 0 & B_{\mu V} & B_{\mu\mu} - \lambda I & B_{\mu p} \\ 0 & 0 & \lambda I & B_{pV} & B_{p\mu} & B_{pp} - \lambda I \end{bmatrix} \begin{bmatrix} \widehat{V} \\ \mu \\ p \\ \overline{V} \\ \overline{\mu} \\ \overline{p} \end{bmatrix} dt. \tag{21}$$

This modified system can then be solved with standard methods to determine the dynamics of an economy under a sticky information structure with a constant learning rate of λ.

With just a few additional lines of code, it is possible to recast a FIRE model into a sticky-expectation environment. Similarly to the FIRE system, the model's jump variables are $\widehat{V}_t$ and p_t, minus whatever predetermined variables are present in p_t. The system therefore satisfies the Blanchard and Kahn (1980) conditions when the number of explosive eigenvalues matches the cardinality of $\widehat{V}_t$ and the non-predetermined variables in p_t.

4 Examples

4.1 A toy representative agent example

My computational approach can be demonstrated using a simple representative agent macroeconomic model that can be exactly solved analytically. Consider the simple FIRE representative agent model with the log-linearized Euler equation:

$$\frac{\mathbb{E}_t[d\widehat{c}_t]}{dt} = \gamma^{-1}\widehat{r}_t$$

where the real interest rate follows $r_t = e^{-\kappa t} r_0$ with r_0 given. Using the households budget constraints and assuming that the household consumption path returns to steady-state, the consumption choice can be written as:

$$c_t = \rho \int_t^\infty e^{-(\tau - t)\rho} \mathbb{E}_t[\widehat{y}_\tau] d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau - t)\rho} \mathbb{E}_t[\widehat{r}_\tau] d\tau. \tag{22}$$

With some calculus and a goods market clearing condition that $y_t = c_t$, the output response to the sequence of real interest rate deviations is

$$y_t = -\gamma^{-1} \frac{1}{\kappa} r_0 e^{-\kappa t}$$

Suppose instead households update their information about the macroeconomic environment at a rate of λ. Aggregate consumption is then chosen in a way that depends on the aggregate expectation $\overline{\mathbb{E}}_t$:

$$c_t = \rho \int_t^\infty e^{-(\tau - t)\rho} \overline{\mathbb{E}}_t[\widehat{y}_\tau] d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau - t)\rho} \overline{\mathbb{E}}_t[\widehat{r}_\tau] d\tau. \tag{23}$$

In the appendix, I show using sequence-space solution techniques that the sticky-expectation law of motion will be

$$\frac{dy_t}{dt} = \left(\frac{d\mu_t}{dt} \frac{1}{\mu_t} + (1 - \mu_t)\rho \right) y_t + \gamma^{-1} \mu_t r_t,$$

where $\mu_t = 1 - e^{-\lambda t}$ is the fraction of households who have updated to full information. In the limit as $\rho \to 0$, the exact closed form solution is:

$$y_t = -\gamma^{-1} \frac{1}{\kappa} \big(e^{-\kappa t} - e^{-(\lambda + \kappa)t} \big) r_0. \tag{24}$$

Using the machinery from the previous section, the A matrix Jacobian for the FIRE system is thus

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{c}] \\ d\widehat{r} \end{bmatrix} = \begin{bmatrix} 0 & \gamma^{-1} \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} \widehat{c} \\ \widehat{r} \end{bmatrix} dt$$

while the B matrix is here identical to the A matrix, as there are no static variables. The new augmented system will be

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{c}] \\ d\widehat{r} \\ d\overline{y} \\ d\overline{r} \end{bmatrix} = \begin{bmatrix} 0 & \gamma^{-1} & 0 & 0 \\ 0 & -\kappa & 0 & 0 \\ \lambda & 0 & -\lambda & \gamma^{-1} \\ 0 & \lambda & 0 & -\lambda-\kappa \end{bmatrix} \begin{bmatrix} \widehat{c} \\ \widehat{r} \\ \overline{y} \\ \overline{r} \end{bmatrix} dt \tag{25}$$

Clearly, the eigenvalues of the system are 0, $-\kappa$, $-\lambda$, and $-(\lambda+\kappa)$. Technically, the system is borderline indeterminate – as the rational expectations model I started with is borderline indeterminate. However, if we require that $\widehat{c}_t$ return to steady-state (and not just remain bounded), then this is a constraint on the zero eigenvector (the nullspace of the matrix). As I show in Appendix A.8, solving for the stable subspace of equation (25) recovers equation (24) for aggregate GDP exactly.

4.2 A Canonical HANK model

In this section, I solve a canonical HANK model with sticky expectations using both my state-space approach and the sequence-space approach of Auclert, Rognlie, and Straub (2020). The model is essentially the one solved in Kwicklis (2025a); the reader should refer to that paper for the model's derivation and details. There are only two important changes. First, while the original model was solved with full information and rational expectations, the model in this section is of course solved with sticky expectations. Second, the calibration in this section uses a more conventional active monetary/passive fiscal form, as opposed to the "active fiscal" experiments considered in Kwicklis (2025a). The central bank raises nominal interest rates more than one-to-one with inflation with a Taylor rule coefficient of 1.5, while the government adjusts taxes over time to slowly stabilize its debt. All other parameters are unchanged and are listed in the appendix.

For illustration, I consider two different shocks: a monetary policy shock (ζ_{mp}) that lowers the interest rate by 1% on impact, and a fiscal transfer shock (ζ_{tax}) that sends flat transfers valued at 1% of annualized steady-state GDP to all households simultaneously. The monetary policy shock demonstrates how the methodology properly leads impulse responses generated by general equilibrium feedbacks to become hump-shaped, while the fiscal transfer shock demonstrates how the model handles instantaneous feedbacks to households' individual budget constraints.

4.2.1 Abridged setup

Households choose consumption c_t and take hours worked L_t as given (chosen by their union to meet aggregate labor demand). They save via an non-contingent bonds a_t. subject to idiosyncratic risk about their labor productivity z_t, which follows a Gaussian log Ornstein-Uhlenbeck process with a mean reversion parameter of θ_z and a variance parameter of σ_z^2:

$$d\log(z_t) = -\theta_z \log(z_t)dt + \sigma_z dW_{z,t}$$

where $W_{z,t}$ is a standard normal Weiner process. The FIRE Hamilton Jacobi Bellman (HJB) equation is

$$\begin{aligned}\rho V_t(a_t, z_t) = &\max_{c_t} \left\{ \frac{c_t^{1-\gamma} - 1}{1-\gamma} - \frac{L_t^{\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \partial_a V_t(a_t, z_t)(r_t a_t + w_t L_t + T_t(z_t, \zeta_t) - c_t) \right\} \\ &+ \partial_z V_t(a_t, z_t) \left(\frac{1}{2}\sigma_z^2 - \theta_z \log z_t \right) + \frac{1}{2}\sigma_z^2 z_t^2 \partial_z^2 V_t(a_t, z_t) + \partial_t V_t(a_t, z_t) \\ &\text{s.t. } a_t \geq 0.\end{aligned} \tag{26}$$

The first-order conditions imply the household chooses $c_t = (\partial_a V_t)^{-1/\gamma}$, such that $h(V) = (\partial_a V)^{-1/\gamma}$. The distribution of households evolves according to

$$\frac{\partial \mu_t}{\partial t}(a, z) = -\frac{\partial}{\partial a}\left(\frac{da_t}{dt}(\overline{V}_t, p_t, a, z)\mu_t(a, z) \right) - \frac{\partial}{\partial z}\left(\frac{\mathbb{E}_t[dz_t]}{dt}\mu_t(a, z) \right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 z^2 \mu_t(a, z) \right). \tag{27}$$

$$\frac{da}{dt}(\overline{V}_t, p_t, a, z) = r_t a_t + w_t L_t + T_t(z, \zeta) - h(\overline{V}_t)$$

Decentalized unions negotiate wages such that wage inflation (and overall inflation, if the passthrough from firms to consumers is complete) abides by a New Keynesian Phillips curve similar to the one in Auclert, Rognlie, and Straub (2024):

$$\frac{\mathbb{E}_t[d\pi_t]}{dt} = r_t \pi_t - \frac{\varepsilon_\ell}{\theta_w}\frac{L_t}{Z}\int\int \left(h_t(a, z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau) z w_t c_t(a, z)^{-\gamma} \right) \mu_t(a, z) da\ dz. \tag{28}$$

The Fisher equation connects $r = i_t - \pi_t$. Tax policy is set via a slow-moving passive rule

$$T_t = \tau w_t L_t + \phi_B(B - B^*) + \zeta_{\text{tax},t}, \tag{29}$$

and government bonds evolve according to

$$\frac{dB_t}{dt} = -(T_t - G_t) + (i_t - \pi_t) B_t. \tag{30}$$

Monetary policy is set with a Taylor rule, plus a monetary policy shock:

$$i_t = r^* + \phi_\pi \pi_t + \zeta_{\text{mp},t}. \tag{31}$$

The aggregate shocks ζ follow a mean-reverting process $d\zeta_{i,t} = -\theta_i \zeta_{i,t} dt$, such that

$$\zeta_{i,t} = e^{-\theta_i t}\zeta_{i,0}. \tag{32}$$

I linearize equations (26-31) around the non-stochastic steady-state wherein the aggregate shocks are disabled: $\zeta_{i,0} = 0$. From there, I solve the linearized FIRE version of the model using the methodology of Bayer and Luetticke (2020) in state-space and using the sequence-space Jacobian (SSJ) algorithm of Auclert et al (2021) in sequence-space. I then solve the sticky expectation variation of the model using my methodology in state-space and the Auclert et al (2020) methodology in sequence-space.

4.2.2 Simulation results

In Figure 1, I depict the impulse response functions of output and inflation to a 1% reduction in the interest rate and a 1% of GDP increase in lump-sum government transfers using the two different

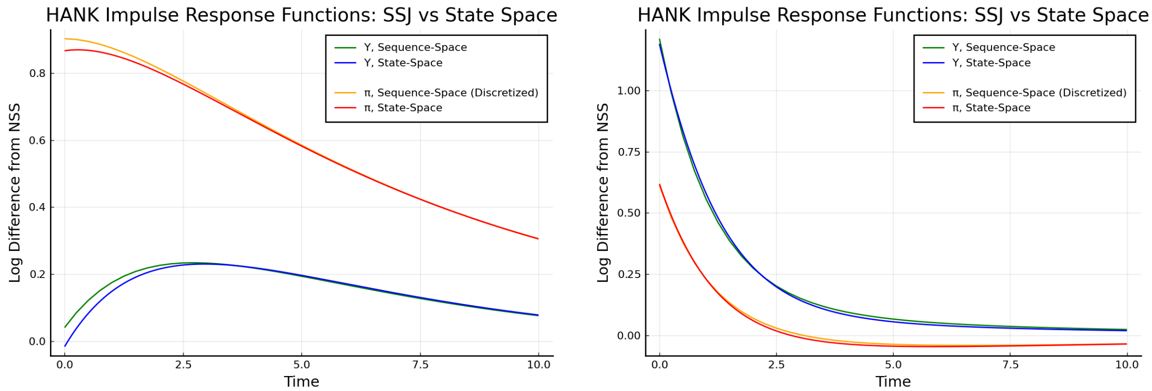

Figure 1: Response of a canonical HANK model to a 1% monetary policy shock and a 1% government transfer shock, as a percentage deviation from the non-stochastic steady-state. Orange and red denote inflation (using the Auclert et al SSJ framework and my state-space approach, respectively). Blue and green denote GDP.

solution methods. Here, I set $\lambda = 0.30$, such that roughly half of the households have fully updated for the presence of the macroeconomic shock 2.5 quarters after the shock's impact.

On impact, a small gap appears on impact between the two impulse response functions. This is because the sequence-space solution becomes higher and higher dimensional in continuous time as the time grid becomes finer, which in turn limits the resolution of the continuous time SSJ solution.[6] Approximation error notwithstanding, the state-space methodology broadly coincides with the sequence-space one, even despite the fact that the state-space approach undergoes dimension reduction with a fixed copula. Note that even though stimulus checks are macroeoconomic variables that only a zero measure of households observe upon impact, the households do immediately observe an influx of resources into their individual idiosyncratic accounts. As such, output jumps on impact. The model is linear with respect to macroeconomic shocks, so if government transfers are reduced aggregate demand also falls on impact, exactly inverting the pattern of a stimulus check disbursal.

As one might expect, increasing the learning rate by increasing λ leads the impulse response functions to a monetary policy shock to more closely resemble those of the FIRE model. This property is displayed in Figure 2. In the FIRE setting, output and inflation jump as soon as interest rates are lowered. In the sticky expectations setting, however, the output response takes more time to build and peaks lower as λ decreases.

5 Conclusion

In this article, I demonstrate how to solve a linearized sticky-expectation HANK model in sequence-space by recycling the Jacobians obtained from the full information, rational expectations version of the system. The approach is simple to implement, and only requires re-arranging the block matrices of the FIRE problem. Each step is justified by relatively intuitive theoretical arguments, which are simplified by working in continuous time and hold for a very broad class of models. I then provide two

[6] Naturally, Auclert, Bardóczy, et al. (2021) originally formulated the SSJ approach for discrete time. Greater numerical accuracy could be obtained by using a non-uniform time step mesh – but this further complicates the approach. As the size of the discretized dt time steps falls, the solution methods align more closely.

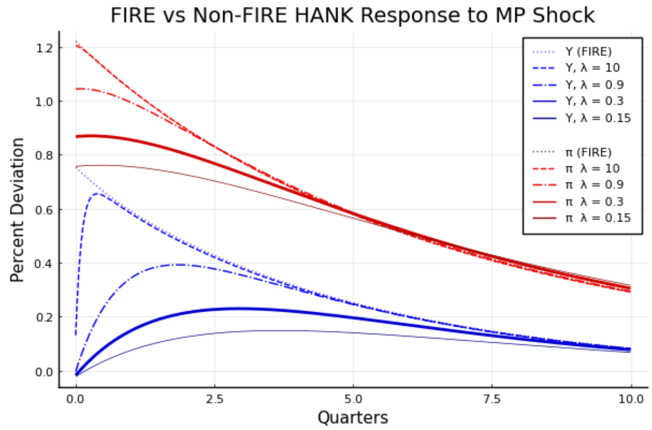

Figure 2: A canonical HANK model's response to a 1% monetary policy shock, for differing degrees of expectation stickyness λ.

concrete applications of the solution technique – a simple analytical one, and a full-fledged numerical HANK model – and show that my methodology produces the correct answer in the first case and closely matches the sequence-space Jacobian numerical approximation in the second.

With this framework developed, a natural next step is to apply it to an empirically interesting problem. I do this in Kwicklis (2025b). Many other interesting extensions exist, however. For instance, because the sticky average beliefs are explicitly tracked in the model, one could consider the effect of "expectations shocks" that directly change average beliefs, but do not directly affect market fundamentals. Additionally, while I only consider the simple sticky expectations case, it may also be possible to incorporate other learning and information structures into the state-space setting, as Bardóczy and Guerreiro (2024) have done in the sequence-space setting. It may also be possible to adapt some parts of the solution technique to discrete time models as well, although proving that the technique works may be more challenging. In any case, this methodology offers an easily implemented tool to allow researchers to consider alternatives to the full information, rational expectations setting – while still maintaining the convenience of a state-space framework.

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A Appendix A: Derivation Proofs

A.1 Proposition 2.1: Derivation of the HJB Equation for i-Belief Households

Statement: *The Hamilton Jacobi Bellman (HJB) equation for $x \in \mathcal{X} \setminus \partial\mathcal{X}$ takes the form*

$$\rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{\tilde{c}_t^i} \Big\{ u(\tilde{c}_t^i) + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \frac{1}{2} tr(\sigma_x(x)\sigma_x(x)' \nabla_x^2 V^i) dt$$
$$+ \nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \tilde{\mathbb{E}}_t^i[g(p, \mu)] + \int_{\mathcal{X}} \delta_\mu V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \tilde{\mathbb{E}}_t^i[\mathcal{D}_t^*(V, p_t)\mu_t] dx \Big\}.$$

Proof. The value function may be additively separated to write

$$\begin{aligned}
\tilde{\mathbb{E}}_t^i \int_t^\infty e^{-(\tau - t)\rho} u(c_\tau) d\tau &= \tilde{\mathbb{E}}_t^i \int_t^{t+dt} e^{-(\tau - t)\rho} u(c_\tau^i) d\tau + \tilde{\mathbb{E}}_t^i \int_{t+dt}^\infty e^{-(\tau - t)\rho} u(c_\tau^i) d\tau \\
&= u(c_t^i) dt + e^{-\rho dt} \tilde{\mathbb{E}}_t^i \underbrace{\tilde{\mathbb{E}}_{t+dt}^i \int_{t+dt}^\infty e^{-(\tau - [t+dt])\rho} u(c_\tau^i) d\tau}_{V_{t+dt}(x_{t+dt}; p_{t+dt}, \mu_{t+dt})} \\
&= u(c_t^i) dt + e^{-\rho dt} \tilde{\mathbb{E}}_t^i V_{t+dt}(x_{t+dt}; p_{t+dt}, \mu_{t+dt})
\end{aligned}$$

So if the subjective measure still obeys the Law of Iterated Expectations (LIE), the discretized HJB is indeed still

$$\begin{aligned}
V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) &= \max_{(c_\tau)_{\tau \geq t}} u(c_t^i) dt + e^{-\rho dt} \tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}, p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) \\
\text{s.t. } \mathbb{E}_t^i[x_t | dW_t] &= \mathbb{E}_t^i f(x_t, c_t^i; p_t) + \sigma_x(x) dW_t
\end{aligned}$$

Approximating $e^{-\rho dt} \approx 1 - \rho dt$,

$$V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{(c_\tau^i)_{\tau \geq t}} u(c_t^i) dt + (1 - \rho dt) \tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}; p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i)$$

Using a Taylor expansion about $dt = 0$,

$$\begin{aligned}
V_{t+dt}^i(x_{t+dt}; p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) =& V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \\
&+ \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \mathbb{E}_t^i[dx_t] + \frac{1}{2} \text{tr}(\sigma_x(x)\sigma_x(x)' \nabla_x^2 V^i) dt \\
&+ \nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \dot{p}_t dt + dt \int_{\mathcal{X}} \delta_\mu V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \partial_t \mu_t dx \\
&+ \nabla_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \dot{p}_t^i dt + dt \int_{\mathcal{X}} \delta_{\mu^i} V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \partial_t \mu_t^i dx \\
&+ \mathcal{O}(dt^2)
\end{aligned}$$

where the Hessian $\nabla^2 V^i$ appears because the differential of the Brownian covariation process $d\langle x \rangle_t$ is proportional to $dW_t^2 = dt$. Taking subjective expectations,

$$
\begin{aligned}
\tilde{\mathbb{E}}_t^i V_{t+dt}^i(x_{t+dt}; p_{t+dt}, \mu_{t+dt}, p_{t+dt}^i, \mu_{t+dt}^i) =& \tilde{\mathbb{E}}_t^i V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) \\
&+ \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)' \mathbb{E}_t^i[dx_t] + \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\
&+ \tilde{\mathbb{E}}_t^i[\nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\dot{p}dt] \\
&+ \tilde{\mathbb{E}}_t^i\left[dt \int_{\mathcal{X}} \delta_\mu V^i(x_t; p_t, \mu_t)\partial_t \mu_t dx\right] + \mathcal{O}(dt^2).
\end{aligned}
$$

where I further assume the orthogonality conditions

$$
\tilde{\mathbb{E}}_t^i[\nabla_p V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\dot{p}] = 0,
$$

$$
\tilde{\mathbb{E}}_t^i V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\partial_t \mu_t = 0
$$

since holding subjective beliefs p_t^i, μ_t^i constant, households in the state-space interior do not forecast their value function to change based on the actual p_t, μ_t.

I can substitute the subjectively expected Taylor expansion back into the HJB to write:

$$
\begin{aligned}
V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{(c_\tau)_{\tau \geq t}} & u(c_t^i)dt + \\
&(1-\rho dt)\Big[V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) + \nabla_x V_t^i(x_t, p_t)'\tilde{\mathbb{E}}_t^i[f(x_t, c_t^i; p_t)dt] \\
&+ \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\
&+ \nabla_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\tilde{\mathbb{E}}_t^i[\dot{p}]dt + dt\int_{\mathcal{X}} \delta_{\mu^i} V^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i\left[\partial_t \mu_t\right] dx + \mathcal{O}(dt^2)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)dt =& \max_{\tilde{c}_t^i} u(c_t^i)dt + \nabla_x V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\mathbb{E}_t^i f(x_t, c_t^i; p_t)dt \\
&+ \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i)dt \\
&+ \partial_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i[\dot{p}]dt \\
&+ \int_{\mathcal{X}} \delta_{\mu^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)\tilde{\mathbb{E}}_t^i\left[\partial_t \mu_t\right] dx + \tilde{\mathbb{E}}_t^i[\partial_t V_t(x_t; p_t, \mu_t)]dt + \mathcal{O}(dt^2)
\end{aligned}
$$

Dividing by dt and taking $dt \to 0$:

$$
\begin{aligned}
\rho V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i) = \max_{\tilde{c}_t^i} \Bigg\{ & u(c_t^i) + \nabla_x V_t^i(x_t; , p_t, \mu_t, p_t^i, \mu_t^i)'\mathbb{E}_t^i[f(x_t, c_t^i; p_t)] + \frac{1}{2}\mathrm{tr}(\sigma_x(x)\sigma_x(x)'\nabla_x^2 V^i) \\
&+ \nabla_{p^i} V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)'\tilde{\mathbb{E}}_t^i[\dot{p}] + \int_{\mathcal{X}} \frac{\delta V_t^i(x_t; p_t, \mu_t, p_t^i, \mu_t^i)}{\delta \mu_t^i(x')}\tilde{\mathbb{E}}_t^i\left[\partial_t \mu_t(x')\right] dx' \Bigg\},
\end{aligned}
$$

as proposed in equation (3).

□

A.2 Sticky mean expectation evolution

Start with the expression for the mean forecast:

$$\overline{\mathbb{E}}_t[Y_{t+s}] = \int_0^\infty \lambda e^{-\lambda\tau} \mathbb{E}_{t-\tau}[Y_{t+s}] d\tau.$$

Differentiating with respect to t,

$$\begin{aligned}
\frac{d}{dt}\overline{\mathbb{E}}_t[Y_{t+s}] &= \frac{d}{dt}\int_0^\infty \lambda e^{-\lambda\tau} \int_\Omega y_{t+s}(\omega)\psi_{t-\tau}(\omega) d\omega d\tau \\
&= \int_0^\infty \lambda e^{-\lambda\tau} \int_\Omega \left[\frac{dy_{t+s}}{dt}(\omega)\psi_{t-\tau}(\omega) + y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt}\right] d\omega d\tau \\
&= \int_0^\infty \lambda e^{-\lambda\tau} \int_\Omega \frac{dy_{t+s}}{dt}(\omega)\psi_{t-\tau}(\omega) d\omega d\tau + \int_0^\infty \lambda e^{-\lambda\tau} \int_\Omega y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt} d\omega d\tau \\
&= \underbrace{\int_0^\infty \lambda e^{-\lambda\tau} \mathbb{E}_{t-\tau}\left[\frac{dy_{t+s}(\omega)}{dt}\right] d\tau}_{\overline{\mathbb{E}}_t[\partial_t Y_{t+s}]} + \int_0^\infty \lambda e^{-\lambda\tau} \int_\Omega y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt} d\omega d\tau
\end{aligned}$$

For the second term, I can interchange the order of integration and integrate by parts with $u = \lambda e^{-\lambda\tau} y_{t+s}(\omega)$ and $v = -\psi_{t-\tau}(\omega)$:

$$\begin{aligned}
\int_\Omega \int_0^\infty \lambda e^{-\lambda\tau} y_{t+s}(\omega)\frac{d\psi_{t-\tau}(\omega)}{dt} d\tau d\omega &= \int_\Omega \int_0^\infty \lambda e^{-\lambda\tau} y_{t+s}(\omega)\left(-\frac{d\psi_{t-\tau}(\omega)}{d\tau}\right) d\tau d\omega \\
&= -\int_\Omega \lambda e^{-\lambda\tau} y_{t+s}(\omega)\psi_{t-\tau}(\omega)\Big|_{\tau=0}^{\infty} d\omega + \int_\Omega \int_0^\infty \psi_{t-\tau}(\omega)\partial_\tau[\lambda e^{-\lambda\tau} y_{t+s}(\omega)] d\tau \\
&= \lambda \int_\Omega y_{t+s}(\omega)\psi_t(\omega) d\omega - \lambda \int_\Omega \int_0^\infty \psi_{t-\tau}(\omega)\lambda e^{-\lambda\tau} y_{t+s}(\omega) d\tau d\omega \\
&= \lambda\Big(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]\Big).
\end{aligned}$$

As such, the complete evolution of the forecast is

$$\frac{d}{dt}\overline{\mathbb{E}}_t[Y_{t+s}] = \overline{\mathbb{E}}_t[\partial_t Y_{t+s}] + \lambda\Big(\mathbb{E}_t[Y_{t+s}] - \overline{\mathbb{E}}_t[Y_{t+s}]\Big).$$

A.3 Proof of 2.3: Characterizing the household choices averaged over beliefs

Proof. Let Δ denote the difference of a variable from its non-stochastic steady-state (when there are no macroeconomic shocks). The consumption choice averaged over all idiosyncratic beliefs will be:

$$\int_i c_t^i(x; p, \mu, p^i, \mu^i) di = \int_i h\big(V_t^i(x; p, \mu, p^i, \mu^i), p_t\big) d\Lambda(i)$$

Let $G_{Vp^i}(x)$ and $G_{V\mu^i}(x)$ denote the Jacobians of V^i with respect to p^i and μ^i at the steady-state, and let $G_{Vp}(x)$ and $G_{V\mu}(x)$ be the Jacobians with respect to the actual p and μ. Additionally, let $\overline{G}_{Vp}(x)$ and $\overline{G}^i_{V\mu}(x)$ denote the Jacobians of the value function calculated with the average beliefs.

Note that since $G_{V\mu}(x)$ is itself a functional derivative, so the operator is in fact shorthand for

$$G_{V\mu}(x)\Delta\mu \equiv D_\mu V[\Delta\mu](x) = \int_{\mathcal{X}} \frac{\delta V(x)}{\delta\mu(x')}\Delta\mu(x')dx.$$

With a first-order Taylor expansion around the steady-state:

$$\begin{aligned}\int_i \Delta c_t^i(x;p,\mu,p^i,\mu^i)di = &\int_i \partial_V h(x)\left[G_{Vp}(x)\Delta p + G_{V\mu}(x)\Delta\mu + G_{Vp^i}(x)\Delta p^i + G_{V\mu^i}(x)\Delta\mu^i\right]d\Gamma(i) + \partial_p h(x)\Delta p\\ &+ \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|)\\ =&\partial_V h(x)\left[G_{Vp}(x)\Delta p + G_{V\mu}(x)\Delta\mu + G_{Vp^i}(x)\Delta\overline{p} + G_{V\mu^i}(x)\Delta\overline{\mu}\right] + \partial_p h(x)\Delta p\\ &+ \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|)\\ =&\Delta h(\overline{V}(p,\mu,\overline{p},\overline{\mu}),p) + \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|)\end{aligned}$$

As such,

$$h(\overline{V}_t, p_t) = c_t(x;p,\mu,\overline{p},\overline{\mu}) = \int_i c_t^i(x;p,\mu,p^i,\mu^i)d\Gamma(i) + \mathcal{O}(\|\Delta p^2, \Delta\mu^2, (\Delta p^i)^2, (\Delta\mu^i)^2\|)$$

in a neighborhood around the non-stochastic steady-state. □

A.4 Proof of Proposition 2.6

Statement: *Suppose the economy starts in its non-stochastic steady state when a macroeconomic shock occurs. If the KFE generator is mass-preserving, then the value function of households at the boundary will satisfy equation (7.5).*

Proof. Let $J(x, c(x), p, \mu)$ denote the probability *flux* vector field for all $x \in \mathcal{X}$, i.e. the instantaneous rate at which probability mass changes in a given direction at a point in the idiosyncratic state-space $x \in \mathcal{X}$ per increment of time. Note that probability mass enters or leaves a region of space in one of two ways: mass flows into the region advectively by being pushed directly by the flows of the mean state equations, or it diffuses out at a rate related to the directional gradient of the mass already in the region relative to surrounding regions. For my problem,

$$J(x, c_t(V_t(x)), p_t, \mu_t) = \underbrace{f(x, c_t(V_t(x)), p_t)\mu_t(x)}_{\text{Advective}} - \underbrace{\frac{1}{2}\nabla_x \cdot (\sigma_x(x)\sigma_x(x)'\mu_t(x))}_{\text{Diffusive}}$$

(Note that $\nabla \cdot A(x)$ is here defined as the gradient operator dotted with each row of the matrix, transforming the matrix into a vector.) The Kolmogorov Forward Equation is equivalent to the statement

$$\partial_t\mu_t(x) + \nabla_x \cdot J(x_t; c(V_t(x)), p_t, \mu_t) = 0.$$

such that the total change in probability density at a point $x \in \mathcal{X}$ is equal to the spatial divergence of the probability flux field. Writing this in terms of the KFE infinitessimal generator,

$$\mathcal{D}^*(V,p)[\mu](x) = -\nabla_x \cdot J(x_t; c_t(V_t(x)), p_t, \mu_t), \text{ such that } \partial_t \mu_t = \mathcal{D}^*(V,p)[\mu_t](x).$$

Note that as the name implies, if the operator is mass-preserving, the total flux of probability through the space $\mathcal{X}$ will always be zero:

$$\begin{aligned}\int_{\mathcal{X}} \partial_t \mu_t(x) &= \int_{\mathcal{X}} \mathcal{D}^*(p,V)[\mu](x)dx = \int_{\mathcal{X}}\int_{\mathcal{X}} D^*(p,V)(x,x')\mu(x')dx'dx \\ &= \int_{\mathcal{X}}\int_{\mathcal{X}} D^*(p,V)(x,x')\mu(x')dx\ dx' = \int_{\mathcal{X}} \underbrace{\left[\int_{\mathcal{X}} D^*(p,V)(x,x')dx\right]}_{0} \mu(x')dx' = 0.\end{aligned}$$

The first equality of the second line above follows from Fubini's theorem, since the boundary is assumed to be rectangular, allowing for the order of integration to be interchanged.

Using the relation between the probability flux field and the infinitessimal generator, it then follows that

$$0 = \int_{\mathcal{X}} \mathcal{D}^*(V,p)[\mu](x)dx = -\int_{\mathcal{X}} \nabla_x \cdot J\ dx^n = -\oint_{\partial\mathcal{X}} J(x) \cdot \vec{n}(x)\ dS$$

where the last inequality follows from Gauss' Divergence Theorem, and S is the boundary of the idiosyncratic state-space (note that the surface integral on the right is one dimension lower than the volume integral on the left). Here, $\vec{n}(x)$ is a unit vector normal to the boundary $\partial\mathcal{X}$, at a point evaluated somewhere along said boundary.

As such, the total net flux of probability mass across the boundary of the state-space must be equal to zero. However, if all of the initial distribution is inside the idiosyncratic state-space (as it is in the non-stochastic steady-state), then this means there can be no flux *anywhere* along the boundary.

Note that the above equality must hold for any distribution, even those that that start with a Dirac delta mass on the boundary. As such, the final integral must hold point-wise. Intuitively, if no mass can cross the $\{x_i = \underline{x}_i\}$ hyperplane, then probability must flow along (tangent to) it. As such, a vector orthogonal to the hyperplane boundary must also be orthogonal to the probability flux:

$$J(x_t) \cdot \vec{n}(x_t) = 0.$$

For a boundary of the form $x_i > \underline{x}$, the orthogonal vector $\vec{n}$ is simply the ith standard basis vector.

Suppose there is no diffusive term for the constrained variable x_i. This then implies if $x \in \partial\mathcal{X}$,

$$J(x; h(V_t(x)), p_t, \mu_t) \cdot \vec{n}(x) = 0$$

$$\iff f_i(x, h(V_t(x)), p_t \mu_t)\mu_t(x) = 0$$

And if $\mu_t(x) > 0$, then

$$\iff f_i(x, h(V_t(x)), p_t) = 0.$$

A mass-preserving KFE infinitesimal generator at each point in time is thus tantamount to a *boundary* *condition*

$$f_i(x, h(V_t(x)), p_t) = 0$$

where $x \in \partial\mathcal{X}$ such that $x_i = \underline{x}_i$.

For example, suppose the law of motion for assets is $f(x_t, c_t, p_t) = r_t x_t + w_t + T_t - c_t$, where $p_t = [r_t, w_t, T_t]$ in this case is the real rate of return, the aggregate wage, and government transfers to households (all macroeconomic objects). The household at the boundary with assets $x = 0$ will then satisfy

$$c_t = h(V_t) = w_t + T_t$$

as an equilibrium condition. This condition is inherited from the fact that the HJB infinitessimal generator is mass preserving – even if the household does not correctly perceive macroeconomic wages and prices. □

A.5 Proof of Proposition 2.7

Statement: *If the KFE infinitessimal generator $\mathcal{D}^*(V, p)$ is a first-order perturbation of the steady-state one with respect to macroeconomic variables, then it will be mass-preserving if the Jacobians evaluated at the steady-state are mass-preserving.*

Proof. This statement is nearly a tautology. To see this, take the kernel and approximate to first order:

$$\partial_t \mu_t(x) = \int_{\mathcal{X}} D^*(V_t, p_t)(x, x')\mu_t(x')dx' \approx \int_{\mathcal{X}} D^*_V(x, x')\Delta V_t(x')dx' + D^*_p(x)\Delta p_t + \int_{\mathcal{X}} D^*_\mu(x, x')\Delta\mu_t(x')dx'$$

Here, D^*_V, D^*_μ, D^*_p are the Jacobians (Frechét in the for μ and V) evaluated in the non-stochastic steady-state. Integrating over the entire distribution, if the Jacobians are all mass-preserving:

$$\begin{aligned}
\int_{\mathcal{X}} \partial_t \mu_t(x)dx =& \int_{\mathcal{X}}\int_{\mathcal{X}} D^*(V_t, p_t)(x, x')\mu_t(x')dx'\ dx \\
\approx& \int_{\mathcal{X}}\int_{\mathcal{X}} D^*_V(x, x')\Delta V_t(x')dx'\ dx + \int_{\mathcal{X}} D^*_p(x)\Delta p_t dx + \int_{\mathcal{X}}\int_{\mathcal{X}} D^*_\mu(x, x')\Delta\mu_t(x')dx'\ dx \\
=& \int_{\mathcal{X}} \left[\int_{\mathcal{X}} D^*_V(x, x')dx\right] \Delta V_t(x')\ dx' + \int_{\mathcal{X}} D^*_p(x)dx\Delta p_t + \int_{\mathcal{X}} \left[\int_{\mathcal{X}} D^*_\mu(x, x')dx\right] \Delta\mu_t(x')dx' \\
=&0.
\end{aligned}$$

If for any distribution μ_t in the approximation,

$$\int_{\mathcal{X}}\int_{\mathcal{X}} D^*(V_t, p_t)(x, x')\mu_t(x')dx'dx = 0$$

then it must also be that up to a first order approximation

$$\int_{\mathcal{X}} D^*(V_t, p_t)(x, x')dx = 0$$

such that the linearized KFE operator will also be mass-preserving in the perturbation if the Jacobians integrate to zero over x. □

A.6 Proof of Proposition 2.8: Average belief value function updating

Statement: *To a first-order approximation, the average belief household updates its value function with a constant factor of* $\lambda(\Delta\widehat{V} - \Delta V)$.

Proof. First, note when an update occurs, the learning occurs to the entire sequence of macro aggregates. At time t, the sequences of beliefs (both about the current state and the future) are updated at rates of $\{\lambda(\mu_{t+s} - \overline{\mu}_{t+s})\}_{s\geq 0}$ and $\{\lambda(p_{t+s} - \overline{p}_{t+s})\}_{t\geq 0}$ multiplied by the time increment dt.

Next, note that the average expectation household and the full information household both essentially solve the same HJB equation:

$$\rho v_t(x_t) = \max_{c_t} \left\{ u(h(v_t(x_t), p_t)) + \nabla_x v_t(x_t)' f(x_t, h(v_t(x), p_t); p_t) \right\} + \partial_t v(x_t)$$
$$\text{s.t. } x_t \geq \underline{x} \ \forall t.$$

which can be linearized such that

$$\partial_t \Delta v_t(x) = \int_{\mathcal{X}} \mathcal{A}_{VV}(x, x')\Delta v_t(x')dx' + \int_{\mathcal{X}} \mathcal{A}_{V\mu}(x, x')\Delta\mu_t(x')dx' + \mathcal{A}_{Vp}(x)\Delta p_t + \mathcal{O}([...]^2).$$

and then – assuming that $\lim_{T\to\infty} \Delta V_T(x) = 0$ – solved forward to write

$$\mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t}) \equiv \int_t^\infty \left[e^{-\mathcal{A}_{VV}(\tau - t)}\right](x, x'') \left[\int_{\mathcal{X}} A_{V\mu}(x, x')\Delta\overline{\mu}_\tau(x')dx' + A_{Vp}\Delta\overline{p}_\tau \right] dx''d\tau.$$

Here, $\mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t})$ represents the solution of the HJB given *sequences* of macro agreggates and distributions from time period t onwards. In equibrium,

$$\Delta\overline{V}_t(x_t) = \mathcal{V}(x, \{\overline{p}_\tau, \overline{\mu}_\tau\}_{\tau\geq t}),$$
$$\Delta\widehat{V}_t(p_t) = \mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t}).$$

Because $\mathcal{V}$ is linear in the macro aggregates, the effect of an update can be expressed as

$$\begin{aligned} &\mathcal{V}(x, \{\lambda(p_\tau - \overline{p})dt, \lambda(\mu_\tau - \overline{\mu}_\tau)dt\}_{\tau\geq t}) = \\ \lambda dt \int_t^\infty \left[e^{-\mathcal{A}_{VV}(\tau - t)}\right](x, x'') &\left[\int_{\mathcal{X}} A_{V\mu}(x, x')[\Delta\mu_\tau(x') - \Delta\overline{\mu}_\tau(x')]dx' + A_{Vp}[\Delta p_\tau - \Delta\overline{p}_\tau] \right] dx''d\tau \\ &= [\mathcal{V}(x, \{p_\tau, \mu_\tau\}_{\tau\geq t}) - \mathcal{V}(x, \{\overline{p}_\tau, \overline{\mu}_\tau\}_{\tau\geq t})]dt \\ &= \lambda[\Delta V_t(x) - \Delta\overline{V}_t(x)]dt. \end{aligned}$$

□

Appendix B: Analytical RANK Example

A.7 Sequence space derivation of a simple RANK model

$$c_t = \rho \int_t^\infty e^{-(\tau-t)\rho} \overline{\mathbb{E}}_t[y_\tau] d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} \overline{\mathbb{E}}_t[r_\tau] d\tau \tag{33}$$

where for the market to clear, $y_t = c_t$ for the representative agent. Suppose monetary policy sets $r_t = e^{-\kappa t} r_0$.

1. Rational expectations: Suppose $\overline{\mathbb{E}}_t = \mathbb{E}_t$. Then if there are no further shocks to the economy,

$$y_t = \rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau$$

where

$$\int_t^\infty e^{-(\tau-t)} r_\tau d\tau = \int_t^\infty e^{-(\tau-t)} r_0 e^{-\kappa\tau} d\tau = r_0 e^{\rho t} \int_t^\infty e^{-(\rho+\kappa)\tau} d\tau = r_0 e^{\rho t} \frac{-1}{\rho+\kappa} e^{-(\rho+\kappa)\tau}\Big|_t^\infty = \frac{r_0 e^{-\kappa t}}{\rho+\kappa}.$$

Meanwhile, setting $G(t) = \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau$,

$$G'(t) = -y_t + \rho G(t)$$

such that

$$0 = -y_t + \rho \underbrace{\int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau}_{G(t)} - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau$$

$$G'(t) - \gamma^{-1} \frac{r_0 e^{-\kappa t}}{\rho+\kappa} = 0.$$

Integrating both sides forward

$$\int_t^\infty G'(s) ds - \gamma^{-1} \frac{r_0}{\rho+\kappa} \int_t^\infty e^{-\kappa s} ds = 0$$

$$\Rightarrow \quad \lim_{s\to\infty} G(s) - G(t) = \gamma^{-1} \frac{r_0}{\rho+\kappa} \frac{1}{\kappa} e^{-\kappa t}$$

$$G(t) = -\gamma^{-1} \frac{r_0}{\rho+\kappa} \frac{1}{\kappa} e^{-\kappa t}.$$

Substituting this into the original expression,

$$y_t = -\rho \overbrace{\gamma^{-1} \frac{r_0}{\rho+\kappa} \frac{1}{\kappa} e^{-\kappa t}}^{G(t)} - \gamma^{-1} \frac{r_0}{\rho+\kappa} e^{-\kappa t} = -\gamma^{-1} \frac{r_0}{\rho+\kappa} \left(\frac{\rho}{\kappa} + 1\right) e^{-\kappa t}$$

$$y_t = -\gamma^{-1} \frac{1}{\kappa} r_0 e^{-\kappa t}$$

2. Sticky info. Suppose a fraction λ updates their beliefs about the macro environment per incre-

ment dt, such that $d\mu_t = \lambda(1-\mu_t)dt$ where the fraction of households who have updated at time t is $\mu_t = 1 - e^{-\lambda t}$. It further follows that $\dot{\mu}_t = \lambda e^{-\lambda t}$. Thus the average expectation is

$$\overline{\mathbb{E}}_t[x_\tau] = (1-\mu_t) \underbrace{(0)}_{\substack{\text{No} \\ \text{update}}} + \mu_t \underbrace{x_\tau}_{\text{Actual}} .$$

Thus

$$\overline{\mathbb{E}}_t[x_\tau] = \mu_t x_\tau$$

Note that $\mu_0 = 0$, $\lim_{t\to\infty} \mu_t = 1$. As $\lambda \to \infty$, $\mu_t \to 1$, while $\lambda \to 0$ causes $\mu_t = 0$.

3. Substituting this into (33),

$$y_t = \mu_t \left[\rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau \right].$$

Differentiating with respect to time,

$$\begin{aligned} \frac{dy_t}{dt} =& \frac{d\mu_t}{dt}\frac{y_t}{\mu_t} + \mu_t \frac{d}{dt}\left[\rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau \right] \\ =& \frac{d\mu_t}{dt}\frac{y_t}{\mu_t} + \mu_t \left[\rho \left(-y_t + \rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau \right) - \gamma^{-1} \left(-r_t + \rho \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau \right) \right]. \\ =& \frac{d\mu_t}{dt}\frac{y_t}{\mu_t} + \mu_t(-\rho y_t + \gamma^{-1} r_t) + \rho \underbrace{\mu_t \left[\rho \int_t^\infty e^{-(\tau-t)\rho} y_\tau d\tau - \gamma^{-1} \int_t^\infty e^{-(\tau-t)\rho} r_\tau d\tau \right]}_{y_t} \end{aligned}$$

Thus

$$\frac{dy_t}{dt} = \left(\frac{d\mu_t}{dt}\frac{1}{\mu_t} + (1-\mu_t)\rho \right) y_t + \gamma^{-1}\mu_t r_t$$

Note that $\frac{d\mu_t}{dt}\frac{1}{\mu_t} = \lambda \frac{(1-\mu_t)}{\mu_t} = \lambda(\mu_t^{-1} - 1)$:

$$\frac{dy_t}{dt} = (1-\mu_t)\left(\frac{\lambda}{\mu_t} + \rho \right) y_t + \gamma^{-1}\mu_t r_t$$

4. In the limit as $\rho \to 0$,

$$\frac{dy_t}{dt} - \frac{d\mu_t}{dt}\frac{1}{\mu_t} y_t = \gamma^{-1}\mu_t r_t.$$

Write an integrating factor as $e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds}$. Multiplying both sides,

$$\underbrace{e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds} \left(\frac{dy_t}{dt} - \frac{d\mu_t}{dt}\frac{1}{\mu_t} y_t \right)}_{\frac{d}{dt}\left[y_t e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds} \right]} = e^{\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds} \gamma^{-1}\mu_t r_t.$$

Solving out the integral, $u = \mu_s$, $du = \frac{d\mu_s}{ds}$,

$$\int_t^\infty \frac{d\mu_s}{ds}\frac{1}{\mu_s}ds = \int_{\mu_t}^1 \frac{1}{\mu_s}d\mu_s = \lim_{s\to\infty}\log(\mu_s) - \log(\mu_t) = -\log(\mu_t)$$

such that

$$\frac{d}{dt}\left[\frac{1}{\mu_t}y_t\right] = \frac{1}{\mu_t}\gamma^{-1}\mu_t r_t.$$

Integrating from t to ∞, assuming $\lim_{\tau\to\infty} y_\tau = 0$:

$$\int_t^\infty \frac{d}{ds}\left(\frac{1}{\mu_\tau}y_s\right)d\tau = \gamma^{-1}\int_t^\infty r_\tau d\tau$$

$$\Rightarrow \quad 0 - \frac{1}{\mu_t}y_t = \gamma^{-1}\int_t^\infty r_\tau d\tau$$

$$y_t = -\gamma^{-1}\mu_t\int_t^\infty r_\tau d\tau$$

If $r_\tau = r_0 e^{-\kappa\tau}$, then

$$y_t = -\gamma^{-1}\mu_t\int_t^\infty r_0 e^{-\kappa\tau}d\tau$$

$$y_t = -\gamma^{-1}\mu_t\frac{1}{\kappa}r_0 e^{-\kappa t}$$

Substituting the definition of μ_t into the expression concludes the derivation.

A.8 Solving the Stable Subspace of Equation (25)

Start with equation (25):

$$\begin{bmatrix}\mathbb{E}_t[d\widehat{c}] \\ d\widehat{r} \\ d\overline{y} \\ d\overline{r}\end{bmatrix} = \begin{bmatrix} 0 & \gamma^{-1} & 0 & 0 \\ 0 & -\kappa & 0 & 0 \\ \lambda & 0 & -\lambda & \gamma^{-1} \\ 0 & \lambda & 0 & -\lambda-\kappa \end{bmatrix}\begin{bmatrix}\widehat{c} \\ \widehat{r} \\ \overline{y} \\ \overline{r}\end{bmatrix}dt$$

The eigenvectors of the system matrix can be collected into the change of bases matrices from (P) and to (P^{-1}) eigen coordinates, where the eigenvector columns correspond to the eigenvalues listed in descending order:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -\kappa\gamma & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -\kappa\gamma & 0 & -\kappa\gamma \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & \frac{1}{\gamma\kappa} & 0 & 0 \\ 0 & -\frac{1}{\gamma\kappa} & 0 & 0 \\ -1 & -\frac{1}{\gamma\kappa} & 1 & \frac{1}{\gamma\kappa} \\ 0 & \frac{1}{\gamma\kappa} & 0 & -\frac{1}{\gamma\kappa} \end{bmatrix}$$

Under the restriction that system dynamics are orthogonal to the zero eigenvector (such that the system strictly returns to steady-state), the first row of P^{-1} dotted with the state system must be zero, such that

$$\widehat{c}_t = -\gamma^{-1}\frac{1}{\kappa}\widehat{r}_t.$$

The updating households will switch to the full information IRF once they become aware of the shock. The choice of y_t will then correspond with the $\overline{y}$ expectations (after being updated for learning). Eliminating the solved consumption choice of the updating households, I arrive at the stable system

$$\begin{bmatrix} d\widehat{r} \\ d\overline{y} \\ d\overline{r} \end{bmatrix} = \begin{bmatrix} -\kappa & 0 & 0 \\ -\lambda\gamma^{-1}\kappa^{-1} & -\lambda & \gamma^{-1} \\ \lambda & 0 & -\lambda-\kappa \end{bmatrix} \begin{bmatrix} \widehat{r} \\ \overline{c} \\ \overline{r} \end{bmatrix} dt$$

With the control variable associated with updating agents substituted out, the system is in its stable subspace; it is simply a system of linear ODEs with a known set of initial conditions. Integrating this system forward (starting with interest rates, then expected interest rates, and then expected output),

$$\begin{bmatrix} \widehat{r}_t \\ \overline{y}_t \\ \overline{r}_t \end{bmatrix} = \begin{bmatrix} e^{-\kappa t} & 0 & 0 \\ -\gamma^{-1}\frac{1}{\kappa}\left(e^{-\kappa t} - e^{-(\lambda+\kappa)t}\right) & e^{-\lambda t} & -\gamma^{-1}\frac{1}{\kappa}\left(e^{-\lambda t} - e^{-(\lambda+\kappa)t}\right) \\ e^{-\kappa t} - e^{-(\lambda+\kappa)t} & 0 & e^{-(\lambda+\kappa)t} \end{bmatrix} \begin{bmatrix} \widehat{r}_0 \\ \overline{y}_0 \\ \overline{r}_0 \end{bmatrix}.$$

Using the initial conditions that $\overline{y}_0 = 0$ and $\overline{r}_0 = 0$,

$$y_t = -\gamma^{-1}\frac{1}{\kappa}\left(e^{-\lambda t} - e^{-(\lambda+\kappa)t}\right)r_0.$$

The linearized solution matches the closed form solution obtained with $\rho = 0$ exactly. Note that $\overline{y}_t$ (post updating) is equal to the actual realized y_t; aggregate output is equal to the consumption decisions chosen by all of the agents in the population, averaged over their beliefs.

B Appendix C: Canonical HANK Model Parameters

The following parameters are used for the numerical solution presented in Section 4.2.

Table 1: Numerical Solution: HANK Model Parameters

Parameter	Symbol	Value	Source or Target
Households			
Internally Calibrated:			
Quarterly Time Discounting	ρ	0.021	$r = 2\%$ Annually
Idiosyncratic Income Shock Variance	σ_z^2	0.017	Floden and Lindé (2001)
Idiosyncratic Shock Mean Reversion	θ_z	0.034	Floden and Lindé (2001)
Assumed from Literature:			
Relative Risk Aversion	γ	2.0	McKay et al (2016)
Frisch Elasticity of Labor	η	0.5	Chetty (2012)
Labor Market			
Labor Elasticity of Substitution	ε_L	10	Philips Curve slope of 0.07
Rotemberg wage adjustment cost	θ_w	100	Philips Curve slope of 0.07
Government			
steady state government debt	B_{NSS}	2.63	HANK $iMPC_0 \approx 0.40$
Geometric maturity structure of debt	ω	0.043	Avg. maturity of 70 months
Income Tax Rate	τ	0.25	
Taylor Rule Coefficient	ϕ_π	1.5	Active monetary policy
Fiscal Debt Coefficient	κ	0.10	Passive fiscal policy
Shocks			
Mean reversion of fiscal shocks	θ_{Tax}	1.0	