Output and Inflation in an Active-Fiscal, Passive-Monetary HANK

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Abstract

When fiscal policy is active and monetary policy is passive in a heterogeneous agent new Keynesian (HANK) model, deficit-financed transfers to poorer households lead to similar amounts of cumulative inflation but greater increases in real output than transfers to wealthier households. Similarly, while the overall impact of monetary policy on the price level aligns with an active-fiscal/passive monetary representative agent benchmark, the presence of low-wealth households who react only to current income attenuates the effect of monetary policy on output. For both monetary and fiscal policy, household heterogeneity is of first-order importance for real variables but has little impact on cumulative inflation.

Keywords: fiscal theory, heterogeneity, inflation, HANK

JEL: E63, E31, E12

1. Introduction

The trade-off between real output and inflation following unanticipated changes to monetary and fiscal policy remains a long-standing open question in macroeconomics. Much of the previous literature has focused on models where monetary policy is "active" and fiscal policy is "passive," in the parlance of Leeper (1991). This paper departs from that assumption and instead explores the implications of active fiscal and passive monetary policy for output and inflation in a canonical heterogeneous agent New Keynesian (HANK) model with idiosyncratic income risk and incomplete asset markets, such that households are heterogeneous in their marginal propensities to consume (MPCs).

The behavior of households with assets is key to determining the path of inflation, while

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the path of output is more strongly influenced by those who do *not* have assets and instead have high MPCs. If the government sends deficit-financed transfer payments to low-wealth households with high MPCs, then the cumulative response of real GDP is predictably larger than when the transfers are sent to wealthier, lower-MPC households. However, the long-term effect on the price level, total cumulative inflation, is largely the same under both transfer policies, provided they lead to similar amounts of nominal government debt (net nominal private assets) that are not paid off by future tax revenue and instead are inflated away. This is contrary to popular intuition that output gaps and the price level move proportionally as measured by "sacrifice ratios" that can ignore household heterogeneity. Instead, different policies present stark differences in their inflation/output trade-offs depending on how they interact with the distribution of households: MPC heterogeneity changes the timing of cumulative output gaps, which changes how they translate into inflation in even the simplest forward-looking New Keynesian Phillips Curve. The trade-off is furthermore very different from the behavior of a HANK under a standard passive fiscal/active monetary regime, where the central bank's target is the key determinant of the path of inflation.

The way that MPC heterogeneity matters strongly for output, but not inflation, has similar implications for a monetary policy shock in an active-fiscal/passive-monetary environment as well. I consider how persistently higher interest rates can temporarily lower inflation through a forward guidance wealth effect on long-maturity bond prices, a mechanism developed in Sims (2011) and Cochrane (2018); my model is the first to extend their monetary policy mechanism to a heterogeneous agent setting. Following an unanticipated change in nominal interest rates, household MPC heterogeneity significantly alters the path of real variables but does little to change the path of inflation relative to what a representative-agent model would produce.

My analysis comes in two parts. First, I outline a two-agent New Keynesian (TANK) model where one group of households smooths consumption with their savings, while the other group is constrained to spending their income as soon as it is received. The price level can be determined either via the fiscal theory of the price level (FTPL), as described at length in Cochrane (2018), or by the demand theory of the price level outlined in Hagedorn (2016) if savers derive utility from holding government bonds; in either case, my main findings are robust. This simplified setting is particularly useful in demonstrating how the main findings pertaining to transfer payments are consistent with the new Keynesian Phillips curve.

In the second part of my analysis, I replace the two-agent block of the model with a continuum of households over asset and income states. Agents face uninsurable idiosyncratic income risk and incomplete markets, yielding a canonical HANK framework with active fiscal policy and passive monetary policy. Unlike in the TANK model, households' income and MPC distributions are not assumed ad-hoc, but are instead calibrated to match empirical microeconomic moments, and the wealth distribution is endogenous. Although more complicated than the TANK setting, this model delivers similar conclusions. However, the added realism of asset and income inequality, precautionary savings motives, and endogenous MPCs makes the setting an ideal "laboratory" with which to examine how active fiscal policy regimes function when fiscal transfers are targeted to one group but not another, or when monetary policy changes nominal interest rates for all.

1.1. Related Literature

My HANK model brings together several highly active areas of macroeconomic research. As alluded to in the introduction, I use the terms "active" and "passive" to describe fiscal and monetary policy in the style of Leeper (1991). "Active" fiscal policy pertains is fiscal policy that does not automatically stabilize a government's real debt to steady-state levels over time for all sequences of the price level. Rather, changes in the price level stabilize real government debt in equilibrium, either through changes in inflation or the real interest rate. This is possible provided that debt is nominal and the central bank does not raise real interest rates in response to inflation – such that monetary policy is "passive." A passive fiscal/active monetary policy regime entails the converse: the government balances its budget to stabilize debt for every possible price level, while the central bank commits to raising real rates in response to inflation.

Most previous incomplete-market HANK models, like those pioneered by McKay et al. (2016), Kaplan et al. (2018), and Auclert et al. (2018, 2023b), and many others, use a passive fiscal/active monetary policy mix unlike the active fiscal/passive monetary one that I explore. Still, these preceding papers also characterize the high and low MPCs that result from ex-post household heterogeneity as key determinants of the response of employment and output to shocks at business cycle frequencies. My heterogeneous agent model's non-stochastic steady-state is particularly reminiscent of McKay et al. (2016), and has a similarly calibrated idiosyncratic income process for the household block. However, Werning (2015) and Acharya and Dogra (2020) note that the cyclicality of income risk is a crucial factor for model dynamics and determinacy, making models highly sensitive to the distribution of corporate profits and taxation over the business cycle. To abstract away from these factors, my baseline specification does not feature cyclical variation in corporate markups or real wages, nor does it feature a government that makes large automatic tax adjustments to balance the budget.

Active fiscal policy with passive monetary policy has also been studied extensively in previous work, but largely in the context of the fiscal theory of the price level (FTPL) with representative agent models and complete markets economies; this body of knowledge is surveyed extensively in Cochrane (2023). As such, most of these previous models do not discuss the way active fiscal policy can engage with economies that feature inequality, idiosyncratic income risk, borrowing constraints, and resulting MPC heterogeneity. I do begin with TANK models to describe the forces at work in the HANK model, however, and so there is some overlap between my paper and Bianchi et al. (2023); they fit a TANK model exhibiting the FTPL as well and find that MPC heterogeneity does little to change the path of prices following a fiscal stimulus relative to a representative agent new Keynesian (RANK) model. However, their paper does not look at targeted stimulus payments made to subgroups of the population as mine does, and sets the fraction of hand-to-mouth households to just 7% of the population, leading their TANK model to have very small MPCs relative to the HANK literature's benchmarks (see Auclert et al. (2018)). They also focus primarily on the FTPL; I consider other active-fiscal price determination mechanisms as well.

Of course, despite these differences, my paper still draws significantly from the FTPL literature. Both Sims (2011) and Cochrane (2018) explain how nominal long-term nominal government bond prices can be depressed by sustained high interest rates, creating a wealth effect that temporarily slows the economy and reduces inflation in a representative agent world, at least before neo-Fischerian effects eventually pull inflation higher. I show that this mechanism moves the price level in nearly exactly largely the same way in a heterogeneous agent economy, but with a slightly different implication for the path of output – a novel result.

At the time of writing, this paper is also the first to study active fiscal policy wherein one group receives transfers and others do not in a fully-fledged HANK model with nominal rigidities. However, Kaplan et al. (2023) has also made the important step of combining an active fiscal/passive monetary policy mix with a setting that includes incomplete markets and household heterogeneity, but no nominal rigidities. In a series of numerical experiments in an endowment economy, they show that a one-time fiscal helicopter drop in a heterogeneous agent fiscal theory setting produces more inflation in the short-run than in the representative agent model, as the transfers change the distribution of risk in the economy by moving resources to households at or near their borrowing constraints. The effect is transitory, however, and over time the price level in the heterogeneous agent model converges to the one-time price level jump experienced in the representative agent one. Most pertinently, whether the transfer is directly targeted to the poor or not plays only a relatively small role in their model's inflation dynamics – a property that I show is preserved in a setting with endogenous demand-determined output and sticky prices.[1] Given the difference in focus, but with the shared interest in describing active fiscal policy in heterogeneous agents settings, my analysis should be read as complementary to theirs.

Kaplan et al. (2023) describe their model as the FTPL combined with heterogeneous agents, but Hagedorn (2024) argues against this interpretation and instead claims that the price level in incomplete market environments can be determined by a "demand theory of the price level" (DTPL), as outlined in Hagedorn (2016). In this alternative interpretation, Hagedorn (2024) the price level and the path of inflation is instead determined so as to equate agents' real asset demand (a downward sloping function of the real interest rate) with

[1]Kaplan et al. (2023) additionally finds that while a heterogeneous agent fiscal theory economy enjoys uniqueness and determinacy when the government runs surpluses in the steady-state, multiple equilibria may emerge when the government runs perpetual deficits and $r < g$. The authors suggest policy rules for eliminating this multiplicity of equilibria and run most of their simulations in an $r < g$ setting, but I consign my own model to a more theoretically conventional environment with positive steady-state primary surpluses and $r > g$.

the net supply of government debt to asset markets, a different equilibrium determination mechanism. As evidence, Hagedorn (2024) notes that the price level in incomplete markets is still determinate even when *both* fiscal and monetary policy are passive. The author further observes that in settings where the real interest rate is endogenous in the steady-state (as in overlapping generations models and incomplete markets ones), multiple different price levels can equate the real value of government debt with the present value of government surpluses, a critique of the FTPL shared by Farmer and Zabczyk (2019).

My analysis concurs with the DTPL view. I check the determinacy of my model with three different measures: Blanchard and Kahn (1980) state-space model eigenvalue counting, along with Onatski (2006) winding number criteria like Auclert et al. (2023a) and Hagedorn (2023). These tests agree that my model is still determinate even when both fiscal and monetary policy are passive, and that the DTPL is determinant of the price level, not the FTPL. However, there are strong similarities between the DTPL and FTPL mechanisms. In a DTPL world, if the price level did not adjust to eventually inflate away new nominal balances following a deficit-financed fiscal stimulus, then the persistent wealth effect of the real assets would lead real consumption and debt to explosively diverge rather than return to steady-state.

Conversely, in the FTPL world, if new government nominal debt is not inflated away, then government debt again rolls over into unsustainable levels, violating households' transversality constraint. In both theories, as Hagedorn (2024) notes, the growth of nominal debt is a "sufficient statistic" for the long-term growth in the price level. In experiments with a TANK model, I argue that a setting where DTPL determines the price level does not look that different from a setting where FTPL determines the price level, nor does the DTPL equilibrium look markedly different when fiscal policy is just barely passive, so long as monetary policy is also passive; inflation's role in stabilizing the real value of nominal assets and liabilities is the key force in explaining how much the price level eventually moves in response to a fiscal shock.

All of my simulations are for certainty-equivalent models using linearized perturbations from a non-stochastic steady-state. The solution method is standard for the TANK models, and uses a state-space model and a Schur decomposition as in Sims (2002). For the HANK model, I use the sequence-space Jacobian technique of Auclert et al. (2021), along with the state-space method of Bayer and Luetticke (2020) (a modification of Reiter (2009)) to check the numerical robustness of my findings. All models are solved using finite difference approximations in continuous time.

2. A Two-Agent New Keynesian (TANK) Model

A two-agent TANK model with a saver household and a spender household is the simplest framework to explore household heterogeneity and its implications for output and inflation. Auclert et al. (2018) and Kaplan and Violante (2018) also note that a TANK model with bonds in the utility function (abbreviated to TANK-BIU) also generates a profile of intertemporal MPCs (iMPCs) that is highly similar to that of a HANK model, as the bond utility term mimics the more complicated precautionary savings motives present in incomplete market models.

Relatedly, but perhaps even more importantly for this paper, Auclert (2018) notes that a bonds-in-the-utility model violates Ricardian equivalence[2] and presents an endogenous relationship between the real interest rate and the path of government debt, allowing the model to display a version of the demand theory of the price level presented in Hagedorn (2016). A TANK-BIU framework thus captures the essential elements of both a HANK model's MPC heterogeneity and price level determination. It is the starting point of my analysis, before I show that its implications are robust to the full-HANK setting.

2.1. Households

Time $t \geq 0$ is continuous, while the measure of households is normalized to 1. A $1 - \mu$ fraction of households behave as savers (labeled "s") and solve an intertemporal optimization

[2] In either case, when agents have more assets and the government has issued more liabilities, households want to consume more, because either they mechanically want to substitute from bonds to consumption in the TANK, or because they feel better insured and want to substitute to more consumption in the HANK.

problem much as a single representative agent would:

$$\max_{(c_{s,t})_{t\geq 0}} \mathbb{E}\int_0^\infty e^{-\rho t}\left[\frac{c_{s,t}^{1-\gamma}}{1-\gamma}-\frac{h_{s,t}^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}+\psi\frac{a_t^{1-\gamma_b}}{1-\gamma_b}\right]dt$$
$$\text{s.t. } \frac{da_t}{dt}=(1-\tau)w_t h_{s,t}+r_t a_t+M_{s,t}-c_t$$

where $c_{s,t}$ is the savers' consumption, $h_{s,t}$ is their hours worked, r_t is the real interest rate (the nominal interest rate i_t minus inflation π_t), $M_{s,t}$ are the lump-sum transfers they receive from the government, and a_t are their real assets. Additionally, ρ is the rate of time discounting, γ is the coefficient of relative risk aversion and η is the Frisch elasticity of labor.

The inverse of γ_b sets the agent's marginal elasticity of utility with respect to holding real liquid assets (which in equilibrium are on net government bonds). If $\psi > 0$, then it is possible for the Hagedorn (2016) DTPL to provide determinacy in the TANK framework, as it does in the HANK world. If I set $\psi = 0$, the DTPL is no longer relevant and determinacy can be ensured by alternative mechanisms, like the FTPL when fiscal policy is active.

Savers in the TANK model thus follow an Euler equation:

$$\frac{\mathbb{E}_t[dc_{s,t}]}{c_{s,t}}=\gamma^{-1}\left(r_t+\psi\frac{a_t^{-\gamma_b}}{c_{s,t}^{-\gamma}}-\rho\right)dt \tag{1}$$

Since they are consumption smoothing and forward-looking, growth in the consumption of wealthy saver agents can be characterized by changes in their real asset position and the equilibrium real interest rate.

The remaining μ measure of households are hand-to-mouth spenders (labeled m) who are exogenously constrained to consume all of their income as soon as it is received. This income is composed of their real wage w_t times hours worked $h_{m,t}$ less a constant income tax rate τ, plus net transfers $M_{m,t}$, such that

$$c_{m,t}=(1-\tau)w_t h_{m,t}+M_{m,t}. \tag{2}$$

While these households are constrained, I assume that their preferences for labor and consumption are the same as those of the savers.

2.2. Firms and Price Setting

Labor is the only production input in the model economy, such that

$$Y_t = L_t, \tag{3}$$

where Y_t is aggregate real output and L_t is the aggregate number of effective hours worked. Final goods firms are perfectly competitive and face no frictions in how they set prices to maximize profits, making wage inflation equal to the final consumption goods' inflation.

Output and employment are demand-determined due to nominal rigidities in the labor market, which are in the style of the decentralized labor union environment of Auclert et al. (2018), which is in turn a modification of Schmitt-Grohé and Uribe (2005). A continuum of decentralized unions hires labor from households and resell it to firms, who differentiate the unions with a constant elasticity of substitution ε_L. Labor supply is demand-determined so that all households work the same number of hours ($h_{i,t} = L_t$), and unions are subject to Rotemberg (1982) nominal wage pricing frictions. The result is a nominal forward-looking wage Phillips curve, which is also the overall Phillips curve in the economy:

$$\frac{\mathbb{E}_t[d\pi_t]}{dt} = \rho\pi_t - \frac{\varepsilon_L}{\theta_w} L_t \left\{ (1-\mu)\left[h_t^{\frac{1}{\eta}} - \frac{\varepsilon_L}{\varepsilon_L - 1} w_t c_t^{-\gamma} \right] + \mu \left[(h_t^m)^{\frac{1}{\eta}} - \frac{\varepsilon_L}{\varepsilon_L - 1} w_t (c_t^m)^{-\gamma} \right] \right\} \tag{4}$$

2.3. Fiscal Policy

The model's fiscal authority collects aggregate taxes (net of transfers) equal to T_t; real government expenditures G_t are included in the following equations for generality, but are set to be zero in equilibrium. The aggregate price level in the economy is p_t. The government borrows using long-term nominal bonds as in Cochrane (2018) by issuing nominal perpetuities $\tilde{B}_t$ at a nominal price of q_t, which pay out exponentially declining coupon payments of $\omega e^{-\omega t}$ per increment of time. As such, ω determines the overall maturity of the government's debt portfolio;[3] as $\omega \to \infty$, government debt becomes instantaneously short-term and must

[3]While this may seem like an arbitrary structure, it can be rationalized by having the government issue debt to maintain an exponentially distributed maturity structure, as shown in Cochrane (2018). From there, one can imagine that government debt is bought by a mutual fund, whose shares are, in turn, owned by households as assets, such that every household effectively owns a representative share of the government's debt portfolio. The details of this are relegated to the appendix.

be rolled over immediately with new bonds (analogous to the continuous-time equivalent of a one-period bond in discrete time), while as $\omega \to 0$, each new bond issued becomes a perpetuity.

The market value of real debt outstanding is

$$B_t \equiv \frac{q_t \tilde{B}_t}{p_t}$$

and, as shown in the appendix and in Cochrane (2018), evolves according to backward-looking equation

$$dB_t = -(T_t - G_t)dt + B_t \left[i_t - \pi_t \right] dt + \frac{d\delta_{q,t}}{q_t} B_t. \tag{5}$$

Here, $d\delta_{q,t} = dq_t - \mathbb{E}_t[dq_t]$ denotes the endogenous expectation error on the nominal price of government debt. Nominal bond prices themselves evolve according to the forward-looking equation

$$\frac{\mathbb{E}_t[dq_t]}{dt} = q_t \left(i_t + \omega - \frac{\omega}{q_t} \right) \tag{6}$$

Notably, since the bonds offer nominal payments, the path of nominal interest rates (and the bond portfolio's maturity structure) determines the evolution of nominal bond prices.

2.3.1. Taxes

As a baseline, the fiscal authority in the model taxes labor income at a rate of τ, such that if total effective labor employment in the economy is L_t and real wages are w_t, total income taxes are $\tau w_t L_t$ per unit of time. Households also receive lump-sum transfers from the government, which aggregate to total lump-sum transfers M_t – such that total tax revenue is

$$T_t = \tau w_t L_t - M_t. \tag{7}$$

In the steady-state, the government balances its budget such that

$$M_{NSS} = \tau w L_{NSS} - r_{NSS} B_{NSS}$$

Outside of steady-state, transfers to households of type i can be written as the sum of a shock to transfers to all agents, plus a shock in transfers to only agents of type i:

$$M_{i,t} = M_{NSS} + 4Y_{NSS} \times \left(\zeta_{\text{All},t} + \frac{1}{\mu_i}\zeta_{i,t}\right) - \kappa_B \left(\frac{q_{NSS}}{q_t}B_t - B_{NSS}\right). \tag{8}$$

μ_i is the share of agents of type i while $\zeta_{i,t}$ is the shock to transfers of type i. The transfer shocks are therefore scaled as a percentage of annual steady-state GDP, and are also scaled by the number of recipients so as to represent the same amount of aggregate transfer spending.

The last term regulates a fiscal rule that determines whether or not fiscal policy is active or passive. These taxes do not adjust due to revaluations of the government debt through changes in q_t, and only respond to changes in debt valued at steady-state prices. If $\kappa_B > r_{NSS}$, then taxes automatically adjust to bring debt back to its non stochastic steady-state, making fiscal policy passive. However, if $\kappa_B < r_{NSS}$, then inflation must be what stabilizes debt, making fiscal policy active. In my baseline scenario, I set $\kappa_B = 0$, rendering fiscal policy unambiguously active.

These transfers aggregate naturally from taxes on saver households $M_{s,t}$ and taxes on spender households $M_{m,t}$:

$$M_t = (1-\mu)M_{s,t} + \mu M_{m,t} \tag{9}$$

2.4. Monetary Block

The central bank directly sets nominal interest rates in the economy according to

$$i_t = r^* + \phi_\pi \pi_t + \zeta_{\text{MP},t} \tag{10}$$

where r^* is the interest rate that would prevail in equilibrium in the absence of any aggregate shocks. In theory, the model can be solved so long as the interest rate rule is "passive," such that $\phi_\pi < 1$. In the baseline specification, I set $\phi_\pi = 0$.

2.5. Market Clearing

Aggregate consumption $C_t = (1-\mu)c_{s,t} + \mu c_{m,t}$ is equal to aggregate output:

$$Y_t = C_t \tag{11}$$

and total hours worked are uniform across households:

$$h_{m,t} = h_{s,t} = L_t. \tag{12}$$

The asset market clears when net private wealth equal to aggregate government debt:

$$a_t = B_t. \tag{13}$$

2.6. TANK Equilibrium

An equilibrium given a sequence of aggregate shocks $(\zeta_t)_{t\geq 0}$ and an initial debt level B_0 is therefore a collection of sequences

$$(c_{m,t}, c_{s,t}, h_{m,t}, h_{s,t}, C_t, L_t, Y_t, B_t, w_t, r_t, i_t, \pi_t, M_{m,t}, M_{s,t})_{t\geq 0}$$

where taking sequences of macro aggregates and prices as given,

i. consumptions choices $(c_{m,t}, c_{s,t})_{t\geq 0}$ are consistent with (2), (1), and the savers' transversality condition $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}[u'(c_{s,t})a_t] = 0$

ii. labor allocations $(h_{t,m}, h_{t,s})$ are consistent with the union rule (12)

iii. inflation π_t is consistent with the unions' maximization problem and resulting wage Phillips Curve (4)

iv. nominal government bond prices $(q_t)_{t\geq 0}$ are consistent with (6)

such that

1. Macro aggregates $(Y_t, C_t)_{t\geq 0}$ are consistent with production (3) and aggregation $C_t = \mu c_{m,t} + (1-\mu)c_{s,t}$
2. real wages w_t are constant and real rates r_t obey the Fisher equation $r_t = i_t - \pi_t$
3. nominal interest rates $(i_t)_{t\geq 0}$ follow the central bank's policy rule (10)
4. Government taxes and transfers across the population and over time $(M_{m,t}, M_{s,t})_{t\geq 0}$ follow the rule (8) and aggregate to M_t and T_t via (9) and (7)

5. Government debt B_t given taxes T_t and real rates r_t evolves according to (5)

6. The asset market clears, as in (13). By Walras' law, this also implies goods market clearing (11).

I assume that aggregate shocks mean-revert at constant rates. As such, they can be written recursively, with the shock of type i at time 0 being given as $\zeta_{i,0}$:

$$d\zeta_{i,t} = -\theta_i \zeta_{i,t} dt \tag{14}$$

or solved forward as a sequence

$$\zeta_{i,t} = e^{-\theta_i t} \zeta_{i,0}. \tag{15}$$

Monetary policy shocks revert at a rate of θ_{MP}, while all fiscal shocks revert at a common rate of θ_{Tax}. The solutions are all for a perfect foresight environment. Once the shock is realized, the transition dynamics are deterministic and known to the agents in the model.

3. TANK Experiments

3.1. Calibration

I calibrate my TANK model largely with parameters that are standard in the HANK literature; these parameters will be re-used for the incomplete markets model. They are displayed in Table 1. I set the amount of steady-state debt to 2.63, roughly 80% of annual GDP, to be consistent with the later HANK model and its target for intertemporal MPCs.

I assume monetary policy shocks have a half-life of 4 quarters. In contrast, the mean reversion of fiscal shocks is made to be much stronger with $\theta_{\text{Tax}} = 1.0$. This is intended to better reproduce the speed with which stimulus checks may be sent out; after 4 quarters, the fiscal shocks almost entirely dissipate. Since the path of the shock in the absence of further perturbations may be described with equation (15), this also means that the cumulative effect of an initial shock of $\zeta_0^{\text{Tax}} = 0.01$ has the interpretation of a 1%-of-annual-GDP disbursal of lump-sum stimulus checks.[4]

[4] For example, if the United States economy in 2019 were to be taken to be the non-stochastic steady-state, this would be a spending program of $210 billion.

Table 1: General Model Parameters

Parameter	Symbol	Value	Source or Target
Relative Risk Aversion	γ	2.0	McKay et al (2016)
Frisch Elasticity of Labor	η	0.5	Chetty (2012)
Idiosyncratic Shock Variance	σ_z^2	0.017	Calibrated
Idiosyncratic Shock Mean Reversion	θ_z	0.034	Calibrated
Labor Elasticity of Substitution	ε_L	10	Philips Curve slope of 0.10
Rotemberg wage adjustment cost	θ_w	100	Phillips curve slope of 0.10
Steady-state government debt	B_{NSS}	2.63	HANK $iMPC_0 \approx 0.50$
Geometric maturity structure of debt	ω	0.043	Avg. maturity of 70 months
Income Tax Rate	τ	0.25	
Mean reversion of monetary shock	θ_{MP}	0.175	4-quarter shock half-life
Mean reversion of fiscal shocks	θ_{Tax}	1.0	

Other parameters are specific to the tank model; these are included in Panel A of Table 2. The columns are grouped by whether the model does not have bonds in the utility function (under the heading "TANK"), or whether the model does have bonds in the utility function ("TANK-BIU"). From there, the models are separated based on their policy regime type: "PF/AM" stands for Passive Fiscal/Active Monetary (the standard New Keynesian regime), "AF/PM" stands for Active Fiscal/Passive Monetary (the FTPL in the model without bonds in the utility function, and the DTPL in the model with bonds in the utility function). "PF/PM" stands for a setting in which both fiscal and monetary policy are passive; the DTPL can still provide a determinate equilibrium in this case, even if the FTPL cannot.

Preferences are calibrated to be consistent with $r = 0.005$ in the non-stochastic steady-state, such that nominal and real interest rates are targeted to 2% annually. This means setting $\rho = 0.005$ for the models where bonds do not appear in the utility function (the first two columns). For the last three columns, where bonds do appear in the utility function, the model is more closely analogous to an incomplete markets HANK model. As such, I set $\rho = 0.023$ to be consistent with my HANK model, and then set $\gamma_b = 2.5$, a value which Kaplan and Violante (2018) note leads TANK models to have similar MPCs to HANK ones. I then adjust ψ to achieve a steady state annual interest rate of 2%. I also assume $\mu = 0.26$ across all of the specifications, such that 26% of households are spenders and 74% are savers,

Table 2: Model Parameters (by Policy Regime and Model Type)

		TANK		TANK-BIU		
			FTPL		DTPL	
	Symbol	PF/AM	AF/PM	PF/AM	AF/PM	PF/PM
Panel A: TANK Parameters						
Quarterly Time Discounting	ρ	0.005	0.005	0.021	0.021	0.021
Share of Spender Households	μ	0.26	0.26	0.27	0.27	0.27
Bond Utility Weight	ψ	0	0	0.25	0.25	0.25
Bond Utility Elasticity	γ_b	-	-	2.5	2.5	2.5
Panel B: Policy Mix						
Auto Fiscal Adj.	κ	0.01	0	0.01	0	0.01
Taylor Rule Coef.	ϕ_π	1.10	0	1.10	0	0

to be consistent with the number of borrowing-constrained households in the full HANK model's non-stochastic steady-state.

For the different policy mixes, I make the policies active fiscal by setting $\kappa = 0$ and passive fiscal by setting $\kappa = 0.01$. I similarly make monetary policy active by setting $\phi_\pi = 1.10$ and passive by setting $\phi_\pi = 0$.

In each of the TANK models and policy regimes, I solve the models using standard state-space techniques, with a Schur decomposition as in Sims (2002). I determine each equilibrium to exist and be unique by establishing that the number of jump variables is equal to the number of explosive generalized eigenvalues.

3.2. TANK Results

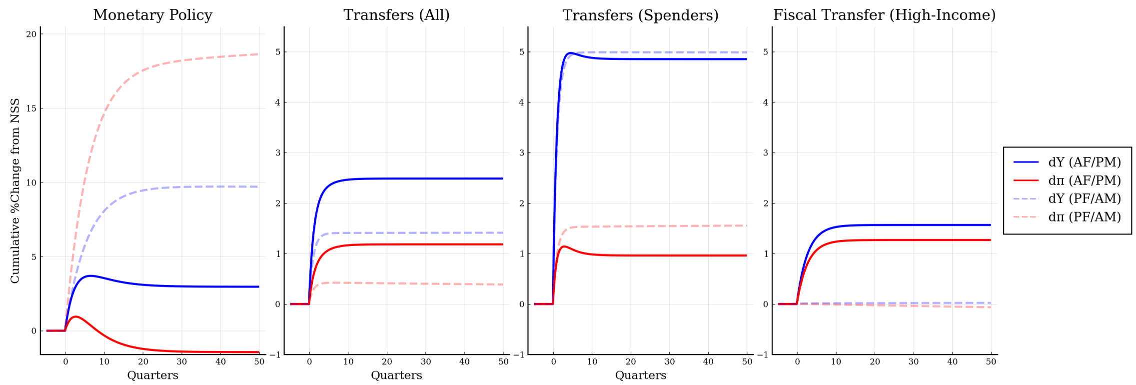
I examine the cumulative effect of shocks on the model economy. As such, I construct ΔY_t^c, the accumulated increase in GDP relative to the non-stochastic steady-state, as

$$\Delta Y_t^c = \frac{1}{Y_{NSS}} \int_0^t (Y_{\tilde{t}} - Y_{NSS}) d\tilde{t} \tag{16}$$

Cumulative inflation $\Delta \pi_t^c$, the total increase in the price level following the shock, can be found by solving the differential equation $\frac{dp_t}{dt} = \pi_t p_t$ forward in time with the initial price level as given:

$$1 + \Delta \pi_t^c = \exp\left(\int_0^t \pi_{\tilde{t}} d\tilde{t} \right)$$

Panel A: TANK



Panel B: TANK-BIU

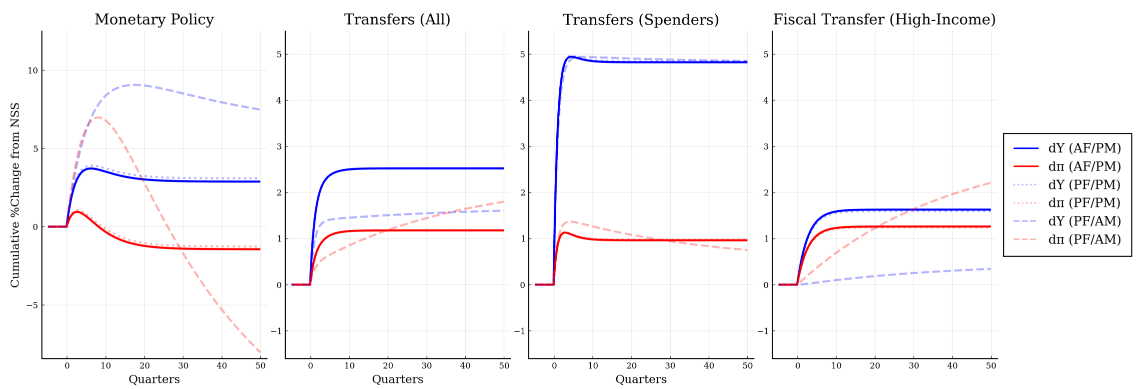


Figure 1: Cumulative impulse response functions in a TANK model, with and without bonds in the utility function (BIU). Shocks include 1% nominal interest rate cuts, and 1% increases in transfers to all agents, spender agents, and saver agents, respectively.

Figure 1 displays the responses of ΔY_t^c and $\Delta \pi_t^c$ following shocks of 1% cuts to interest rates and 1% increases to transfers. Panel A refers to the TANK model without bonds in the utility function ($\psi = 0$), while Panel B includes bonds in the utility function ($\psi = 0.25$). Blue lines refer to the change in real GDP, while red lines refer to the change in inflation. Solid lines refer to the active fiscal/passive monetary mix, while dashed lines refer to the more conventional passive fiscal/active monetary one. Dotted lines refer to the setting where both types of policy are passive; these series are displayed only for the TANK-BIU model since the model is still determinate via the DTPL but not via the FTPL.

For both the TANK and TANK-BIU models, output and inflation dynamics are very similar so long as fiscal policy is active. The accumulated output gap reaches about 2% following a 1% of GDP disbursal of deficit-financed transfers to all households, nearly 5%

following a disbursal to zero-wealth spender households, and about 1.5% following a disbursal to wealthy savers. Inflation, in all of the fiscal transfer scenarios, accumulates just to a little over 1.1%. It barely matters if fiscal policy is also made slightly passive in the DTPL environment if the speed of the fiscal adjustment is slow and monetary policy remains passive; the dotted lines are essentially on top of the solid ones in Panel B, to the point that they require close inspection to see.

If monetary policy is active and fiscal policy is passive, however, the total magnitude of the deficit-financed transfers is no longer sufficient to qualitatively characterize the amount of inflation. Transfers to wealthy savers barely move output and inflation in the baseline TANK (panel A), as the savers are Ricardian and know i) real rates will rise in response to inflation and not fall to erode away government debt, and ii) debt today necessitates fiscal readjustments and taxes in the future.[5] Conversely, transfers to poorer agents imply both more output and more inflation relative to transfers to the savers, as opposed to similar amounts of inflation and more output when monetary policy is passive.

Monetary policy is also less powerful in the passive monetary world than the active monetary one. The "stepping-on-a-rake" effect of Sims (2011) and Cochrane (2018) is apparent in the first graphs of both panels A and B; lower interest rates eventually pull down the price level, too, even though the Cochrane (2018) forward guidance mechanism through long-term bond prices raises short-term inflation in both the FTPL (TANK) and DTPL (TANK-BIU) settings.

In contrast, when monetary policy is active, output and inflation rise far more in Panel A and B. In Panel A, there is no stepping-on-a-rake. In panel B, inflation eventually turns to deflation following the expansionary monetary policy shock, and the output gap eventually turns negative. Still, the dynamics are much more persistent than they would be under different policy regimes. This is because after an unforeseen rate cut, the inflation leads to nominal and real rate increases via the Taylor rule, crashing asset prices and leading spender households to want to rebuild the value of their bond holdings with more savings, reducing aggregate demand; equilibrium is reached when the path of output prices falls by enough to

[5]The slight changes in output and prices that do occur stem from the fact that the wealthy households know that the poor spender households pay part of the future budget-balancing taxes.

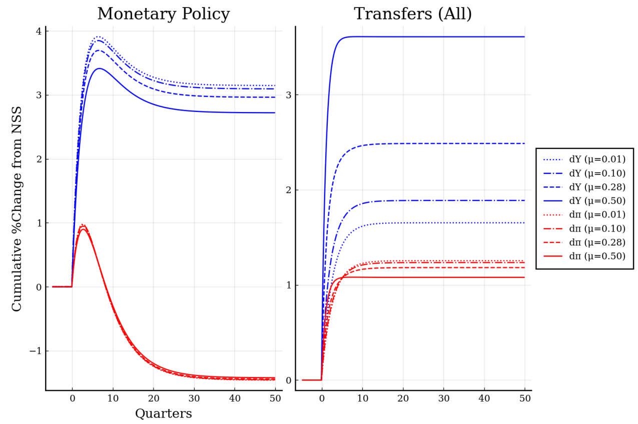


Figure 2: Cumulative responses of output (blue) and inflation (red) in an active fiscal-passive monetary TANK, by the share of population hand-to-mouth. The monetary policy shock is a 1% cut to interest rates, while the transfer shock is a 1% of annual GDP transfer to all agents.

make the asset market clear in each period.

In settings where monetary policy is passive, MPC heterogeneity does little to alter the path of inflation in response to an interest rate cut in the baseline TANK model. Still, it does significantly change the dynamics of real output. The first plot in Figure 2 depicts the responses of real output and the price level to a 1% fall in interest rates, but for economies where spenders make up different shares of the population. Varying the share of hand-to-mouth agents from $\mu = 0.01$ to $\mu = 0.5$, the resulting paths of the price level (in red) are almost indistinguishable. With the Cochrane (2018) mechanism, the price level adjusts to make the marginal bond buyer indifferent to purchasing nominal bonds or output goods following a fall in bond prices. Spender households are not marginal bond buyers, while saver households all behave the same; as a result, the amount of inflation does not change much when the share of constrained households is adjusted.

For real output, though, the MPC heterogeneity does matter, as depicted in the blue lines in the first plot of Figure 2. When more households are constrained to be spenders, fewer of them respond to the changes in interest rates and asset prices, as they only respond to changes in their income. They propagate the general equilibrium effects of the fall in asset prices triggered by a rise in the policy rate but do not respond to it directly, unlike their saver counterparts. As such, as μ increases, the rise in output induced by expansionary monetary policy falls.

In the second plot of Figure 2, I repeat the experiment of varying μ in the economy and studying the responses of output and inflation, but now for a 1% transfer shock to all agents. Once again, the MPC heterogeneity matters significantly for the path of real GDP but less for the path of the price level. In both the DTPL and the FTPL, nominal government debt serves as a nominal anchor for inflation when monetary policy is passive. In FTPL, if the government unexpectedly signals that it will begin running deficits, then households will not want to hold government bonds unless either i) the price level rises immediately or ii) inflation drives down the real interest rate over time to make the present value of future surpluses equal to the current bonds outstanding. If the price level follows any other path, debt grows explosively and violates the savers' transversality constraint.

Similarly, for the DTPL, more real bonds generate a liquidity or wealth effect for households in partial equilibrium, leading them to want to save less and spend more. Not everyone can be a net bond spender in general equilibrium, however, as every seller necessitates a buyer; the price level has to rise to devalue the bonds and clear the asset market, either immediately or over time. If it did not, then to incent households to hold higher real balances in the current period, the return to doing so would have to rise, which would again lead to a divergent path of debt, ruled out as an equilibrium by the transversality condition.[6] If inflation initially overshoots the amount of inflation required to control the ratio of debt to GDP, as it does in the preceding fiscal simulations displayed in 1, then households react to the erosion of their assets by trying to re-accumulate them through saving, instigating a gradual recession and deflation in equilibrium.

The mechanisms are distinct but similar in their implications for fiscal policy. Nominal debt determines the path of prices. MPC heterogeneity changes the way nominal debt can change real employment and output. Different redistribution policies can thus lead to different ratios of inflation to output.

3.3. How is this consistent with the Phillips Curve?

The Phillips curve is generally understood as a dynamic relationship between inflation and the output gap. Equation (4) is slightly more complicated; wage inflation is driven by

[6] Although note that a transversality condition and infinitely-lived households are not required for the DTPL; see Hagedorn (2024), where the DTPL pins down the price level in an overlapping generations setting.

the average wedge between households' marginal utility of leisure and their marginal utility of working.[7] However, because these quantities themselves are closely related to the output gap, the main dynamics of the model are largely unchanged if the Phillips curve is altered to something like

$$\frac{\mathbb{E}_t[d\pi_t]}{dt} = \rho\pi_t - \nu\widehat{Y}_t \tag{17}$$

where $\widehat{Y}_s$ is the output gap and ν is the slope of the Phillips curve, which is larger when firms find it easier to change their prices, and smaller otherwise. Even in this simplified world, the relationship between total *cumulative* inflation and output depends on the timing of the output gaps.

Suppose a shock leads to a sequence of output gaps $(\widehat{Y}_t)_{t\geq 0}$ such that the accumulated output gap asymptotically approaches its terminal level of ΔY_t^c at an exponential rate of α, much as it (roughly) does in the fiscal policy experiments of Figure 1. If the rate of time discounting is not too large, and if the cumulative amount of inflation is relatively small, it follows that the resulting cumulative inflation per accumulated percentage points of output gap can be approximated as

$$\frac{\Delta\pi_\infty^c}{\Delta Y_\infty^c} \approx \frac{\nu}{\alpha}$$

The steps to arriving at this approximation are straightforward and provided in the appendix. When it is easy for agents to change prices, the amount of cumulative inflation relative to output increases, all else equal; when the cumulative output gap converges on its final value very quickly, the amount of relative inflation is smaller.

This intuition is true for a very general variety of forward-looking Phillips curves. It is true that inflation at time t jumps higher when current and future output gaps jump higher. However, if firms or workers and unions take time to adjust their prices, then they are limited in how much they can immediately raise their prices in response to an acute surge in output. Additionally, they are forward-looking, so past output and inflation are sunk; only future

[7] Households like the spenders who have high marginal propensities to consume would like to substitute their consumption for more leisure in equilibrium after receiving transfers. In a world with nominal output price rigidities, the real wage rate would have to rise to get them to keep supplying labor. In the sticky-wage world, decentralized unions bargain for higher wages in the future. When all unions do this at the same time, the nominal wage rises only for the increases to be passed into output prices, leaving the real wage unchanged.

output gaps matter for how they set prices. If real GDP returns to its steady-state value quickly, these future output gaps may be small, even if past output gaps have been large. In this sense, price-setters in the economy tend to fall "behind the curve" for the transitory-but-potent real GDP expansions that transfers to high hand-to-mouth agents generate. By the time the economy returns to steady state, cumulative real output can rise higher for the same rise in the price level when it rises faster.

4. A Heterogeneous Agent New Keynesian (HANK) Model

4.1. Households

I now replace the two-agent household block of the model and replace it with a heterogeneous agent one. Households exist in a Bewley-Aiyagari setting where they have two dimensions of ex-post heterogeneity: their labor-augmenting productivity z (generating income inequality) and their real asset position a (which agents endogenously determine based on their consumption choices). For convenience, I write a as assets valued at steady-state bond prices q_{NSS}, such that $\frac{q_t}{q_{NSS}}a_t$ is a household's real wealth at time t. If $V_t(a,z)$ is a household's value function at time t given their asset position a at steady-state bond prices and labor productivity z, the household problem is

$$
\begin{aligned}
V_t(a_0, z_0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{h_t(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] dt \\
\text{s.t. } \frac{q_t}{q_{NSS}}\frac{da_t}{dt} + \frac{dq_t}{dt}\frac{1}{q_{NSS}}a_t = (1-\tau)w_t z_t h_t(a,z) + r_t\frac{q_t}{q_{NSS}}a_t + M_t(z_t;\zeta_t) - c_t \\
d\log(z_t) = -\theta_z \log(z_t)dt + \sigma_z dW_{t,z} \\
a_t \geq 0.
\end{aligned}
$$

Here, W_t is a classical Weiner process (Brownian motion), such that log labor income follows an Ornstein-Uhlenbeck process in the non-stochastic steady-state. Note that agents do not have bonds in their utility function – but they do value bonds as a means to smooth consumption, particularly in the face of idiosyncratic shocks to their income and a borrowing limit that prohibits their assets from becoming negative. The left-hand side of the consumer's budget constraint represents the value of new assets purchased plus the capital gain associated with existing assets held relative to their steady-state values. The right-hand side represents income plus returns net of consumption (savings plus real returns inclusive of capital gains). Like the TANK households, the agents can receive transfers from the government $M_t(z_t, \zeta_t)$ that depend on where they are in the joint distribution of assets and incomes. However, I restrict the transfers to be contingent upon household income, not assets.

Transfers can either be made to those below median z (denoted $z_{0.50}$), above median z, or to all households:

$$M_t(z, \zeta_t) = 4Y_{NSS} \times \left(\zeta_{\text{ALL},t} + \frac{1}{0.5}\mathbf{1}\{z \leq z_{0.50}\}\zeta_{\text{BELOW},t} + \frac{1}{0.5}\mathbf{1}\{z > z_{0.50}\}\zeta_{\text{ABOVE},t} \right)$$

where $\zeta_{ALL,t}, \zeta_{BELOW,t}, \zeta_{ABOVE,t}$ are aggregate shocks that follow (15).

The household's problem can be recursively formulated as a Hamilton Jacobi Bellman (HJB) equation:

$$\begin{aligned} \rho V_t(a, z) = &\max_c \Bigg\{ \left[\frac{c^{1-\gamma}}{1-\gamma} - \frac{h_t(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \\ &+ \frac{\partial V_t}{\partial a}(a,z)\frac{q_{NSS}}{q_t}\left[(1-\tau)w_t z h_t(a,z) + M_t(z_t;\zeta_t) - c + \left(r_t - \frac{dq_t}{dt}\frac{1}{q_t}\right)\frac{q_t}{q_{NSS}}a\right] \\ &+ \frac{\partial V_t}{\partial z}(a,z) z \left[\frac{1}{2}\sigma_z^2 - \theta_z \log(z)\right] + \frac{\partial^2 V_t}{\partial z^2}(a,z)\frac{1}{2}\sigma_z^2 z^2 + \frac{\partial V_t}{\partial t}(a,z) \Bigg\}. \end{aligned} \tag{18}$$

where households take the path of prices w, r, and q as given, and subsumed into the time subscript of the value functions.

The distribution of households over idiosyncratic states is $\mu_t(a, z)$; it evolves according to the standard Kolmogorov Forward Equation (KFE)

$$\frac{\partial \mu_t}{\partial t}(a,z) = -\frac{\partial}{\partial a}\left(\frac{da_t}{dt}\mu_t(a,z)\right) - \frac{\partial}{\partial z}\left(\frac{\mathbb{E}_t[dz_t]}{dt}\mu_t(a,z)\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 z^2 \mu_t(a,z)\right) \tag{19}$$

4.2. Wage Phillips Curve

The wage Phillips curve becomes

$$\frac{\mathbb{E}_t[d\pi_t^w]}{dt} = r_t \pi_t^w - \frac{\varepsilon_\ell}{\theta_w}\frac{L_t}{Z}\int\int \left(h_t(a,z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau) z w_t c_t(a,z)^{-\gamma} \right) da\ dz \tag{20}$$

4.3. Market Clearing

As in the TANK model, and as in Auclert et al. (2018), all households supply labor according to a uniform rule: $Zh_t(a,z) = L_t$, where $Z = \int\int z\mu_t(a,z)\ da\ dz$ is average labor productivity.

For the asset market to clear,

$$\frac{q_t}{q_{NSS}} \int\int a\mu_t(a,z) da\ dz = B_t \tag{21}$$

With Walras' law, the consumption market also clears: $Y_t = C_t = \int\int c_t(a,z)\ da\ dz$. Total transfers are similarly $M_t = \int\int M_t(z)\ da\ dz$.

The rest of the model (the fiscal and monetary authority, the evolution of bond prices, and firms) are exactly the same as in the two-agent framework.

5. HANK Model Calibration

The HANK model uses the same parameters as the ones used by the TANK model in Table 1. As in McKay et al. (2016), I additionally calibrate the continuous time income process parameters (θ_z, σ_z^2) via simulated method of moments to match the Floden and Lindé (2001) estimates of the permanent component of annual wage autocorrelation and autoregression variance, residualized for age, occupation, education, and other covariates. I similarly calibrate the time discounting parameter ρ to match a real interest rate of 0.5% quarterly, or roughly 2% annually. These coefficients are all listed in Panel A of Table 3. I solve for the model's non-stochastic steady-state using the methods outlined in Achdou et al. (2021); select moments from this distribution are reported in Panel B of Table 3.

The marginal distributions of households along assets and incomes are displayed in Figure 3. Since the distribution of assets contains an atom at the borrowing constraint, I display the cumulative stationary distribution of assets, followed by the probability density of household incomes. The last plot in Figure 3 depicts the aggregate intertemporal MPCs of households in the non-stochastic steady-state in response to a year-long transfer that integrates to 1. The iMPCs are aggregated to the annual level to make them comparable with Figures 1 and 2 of Auclert et al. (2018). Households in my model spend roughly 43% of the value

Table 3: HANK Model Parameters and Non-Stochastic Steady-State Statistics

Description	Symbol	Value	Source or Target
Panel A: HANK Parameters			
Quarterly Time Discounting	ρ	0.021	$r = 2\%$ Annually
Idiosyncratic Income Shock Variance	σ_z^2	0.017	Floden and Lindé (2001)
Idiosyncratic Shock Mean Reversion	θ_z	0.034	Floden and Lindé (2001)
Panel B: HANK NSS Statistics			
Contemporaneous iMPC (Annual)		0.43	
Debt to Annual Income	$B_{NSS}/(4Y_{NSS})$	0.67	
Correlation btw. Income and Assets	$\text{Corr}(a, z)$	0.56	
Share of households with $a = 0$	$\int \mu_{NSS}(0, z)dz$	0.27	
Asset Gini Coefficient		0.75	
Income Gini Coefficient		0.31	

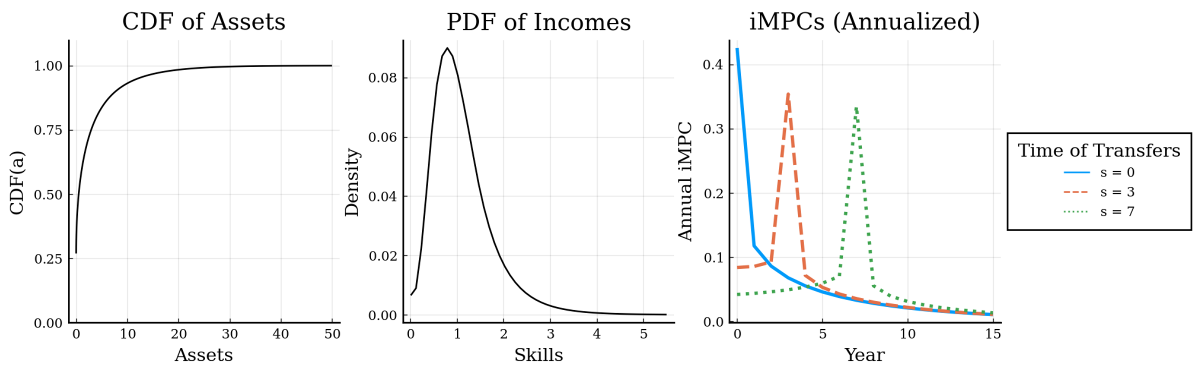


Figure 3: Marginal distributions and intertemporal propensities to consume in the non-stochastic steady-state

of their initial transfer income in the first year when they receive it, 12% a year later, 9% two years later, 7% a year after that, and so on. These iMPCs are roughly consistent with the lower bound presented in Auclert et al. (2018), which uses data from the Italian Survey of Income and Wealth. The plot's dashed lines indicate households' aggregate propensity to spend when a transfer is announced 3 and 7 years in advance; the tent-shaped spending patterns are again reminiscent of Auclert et al. (2018).

5.1. A Note on the Numerical Solution

I use the sequence-space Jacobian method of Auclert et al. (2021) to generate perfect-foresight model solutions to a time zero perturbation of my aggregate shock vector ζ. To check the validity of my numerical solutions, I compare them to a state-space solution of the same model, constructed using the methodology of Bayer and Luetticke (2020). As shown in the appendix, although the methods differ in how they approximate the solution (the latter

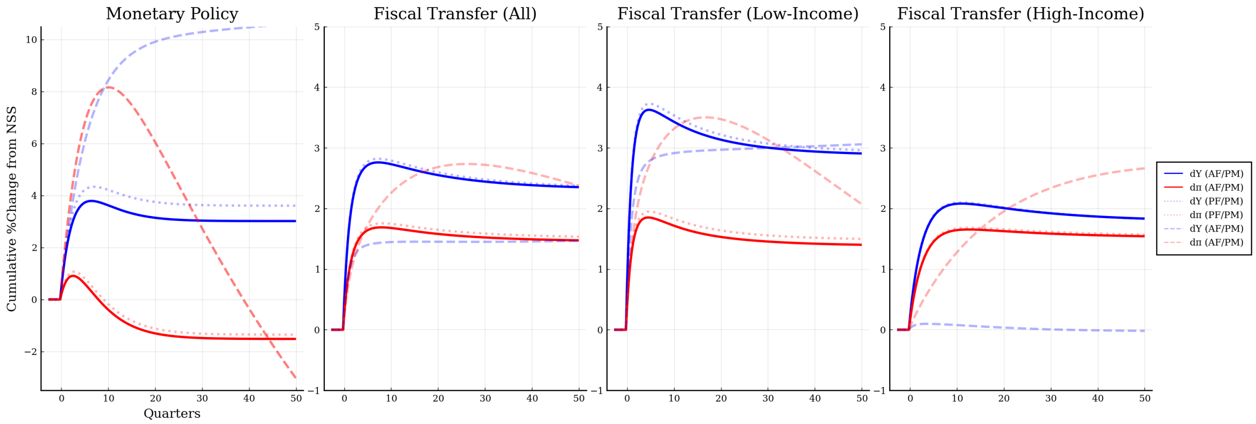


Figure 4: Cumulative impulse response functions in a HANK model. Shocks include 1% nominal interest rate cuts, and 1% increases in transfers to all agents, below-median income agents, and above-median income agents, respectively.

accomplishes dimension reduction using a discrete cosine transformation and a copula, while the former artificially truncates the sequence space after 300 quarters), they yield nearly identical trajectories for output and inflation for all of the shocks in the baseline model.

6. HANK Results

The paths of aggregate cumulative output and inflation in the HANK model following shocks are displayed in Figure 4; the unaccumulated impulse response functions used to create the graph are displayed in Figure 5. Although the model allows for richer heterogeneity in the agents, endogenizes the distribution of MPCs, and features precautionary savings amongst households, the impulse response functions display the same qualitative patterns as the TANK ones. When transfers are sent out to low-income agents with fewer assets and higher MPCs, the output response is larger; the inflation response is largely the same regardless of the distribution of recipient households.

For monetary policy, the Cochrane (2018) mechanism delivers almost the same path of equilibrium prices in a HANK model as it would in a RANK model in an active-fiscal/passive-monetary mix, as exhibited in the first graph of Figure 6. A persistent 1% hike in nominal interest rates generates a temporary deflation, as the path of expected higher rates causes nominal asset prices to fall, lowering the aggregate demand of bond buyers and driving goods prices down until the households are all indifferent to spending and saving. In the HANK model, however, the marginal bond's buyer is relatively wealthy and behaves much like the

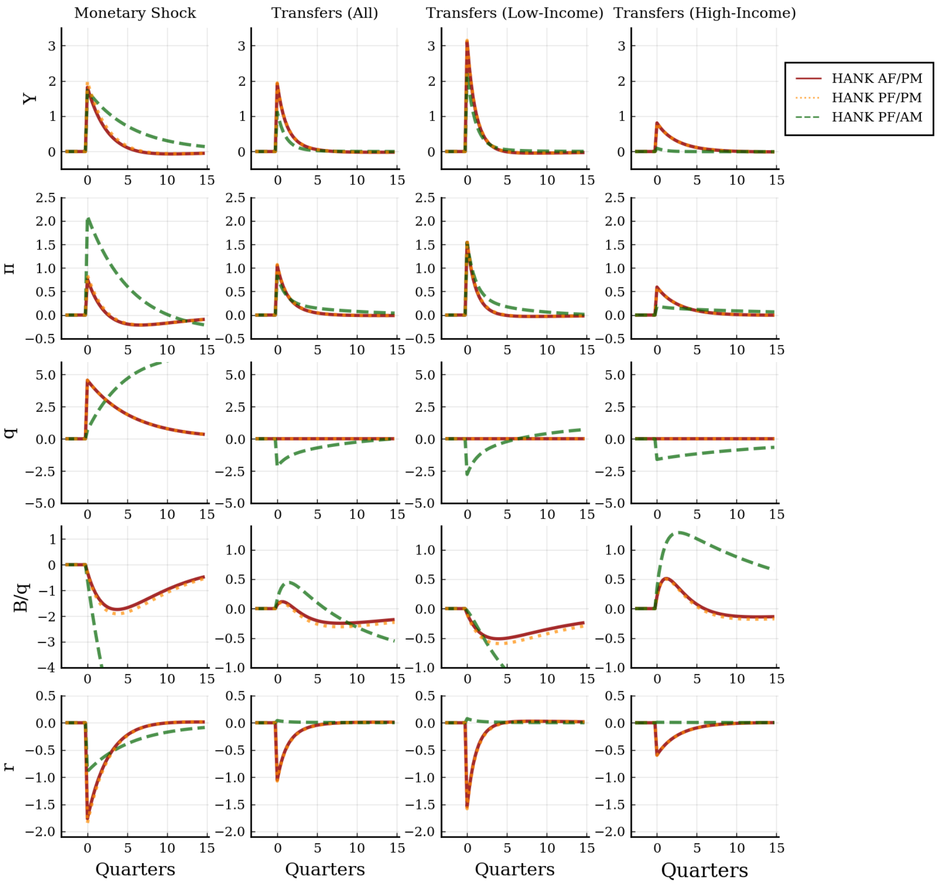


Figure 5: Impulse response functions (unaccumulated) to policy shocks in the HANK environment. Solid lines correspond to the HANK economy under an active-fiscal/passive monetary policy regime. In contrast, dashed lines refer to the economy under a slow fiscal adjustment passive-fiscal/active monetary one, as parameterized by Panel B of Table 2. Dotted lines correspond to a passive-fiscal/passive monetary policy mix.

representative agent – such that the adjustment of the price level nearly matches that of RANK. This is true even though the HANK model's price level is determined by the DTPL, and the RANK's by the FTPL. The high-MPC households, however, largely do not hold bonds and have inelastic bond demand; they do not propagate the wealth effect of monetary policy in the first round, muting its impact on real output.

If transfers are cut to all households uniformly in the economy, fiscal policy is propagated in the first round by low-wealth, high-MPC households. The resulting recession from a transfer cut is worse in the HANK than in the RANK, as shown in the second plot of Figure 6. The path of the fiscal deflation does vary across the two models more than it does for monetary policy, as the fiscal policy interacts with the MPCs more directly than monetary policy, which changes the timing of the output and inflation responses. Even so, the amount

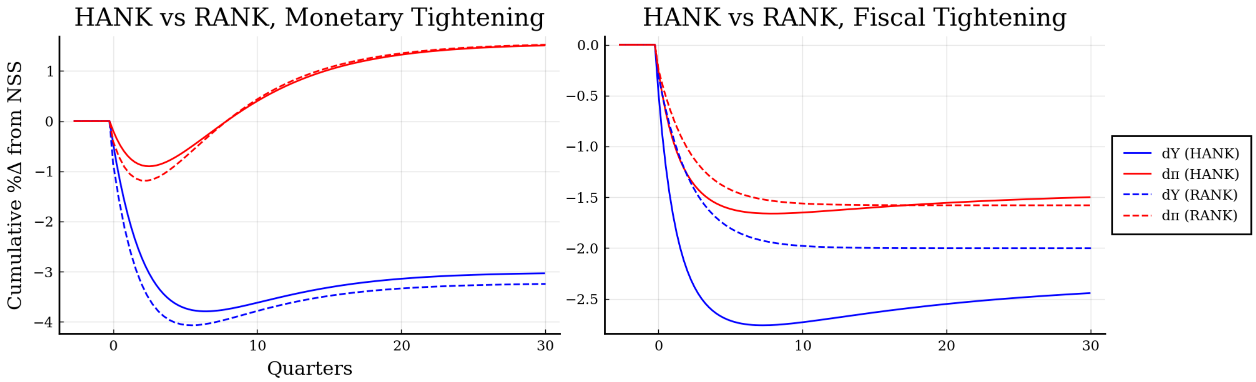


Figure 6: Cumulative impulse response functions to a 1% monetary tightening shock (left) and a 1% of GDP increase in taxes (right), for both a HANK model and a RANK model with active fiscal policy and passive monetary policy.

of cumulative inflation in the two settings ends up being similar in both HANK and RANK; the 1% transfer cut leads to a cumulative -1.5% fall in the price level after 30 quarters, with most of the deflation being realized in the first two years.

When transfers rise, or interest rates fall, what drives the expansion of real output in the active-fiscal/passive-monetary HANK model? The answer is predominantly "direct" effects, i.e., households responding to lower interest rates and higher asset prices for monetary policy and higher income following transfers from fiscal policy. This can be observed in Figure 7, which decomposes the real GDP response of the heterogeneous households into three effects: the effect of real interest rates (and nominal bond prices) (in red), the effect of transfers (in yellow), and the effect of changes in labor demand (in blue). The paths of each of these inputs, determined in equilibrium, are taken as given by households; the colored regions of the plot depict how each contributes to the total movement of real GDP, which is depicted in the black dashed line.

Following a monetary shock with the Cochrane (2018) mechanism, most (85%) of the GDP response of households is attributable to the movement of the real interest rate and bond prices; only 15% of the jump in GDP on-impact is attributable to general equilibrium labor market effects, given fiscal transfers do not automatically adjust. Similarly, when transfers are sent out to the general population and to low-income households in particular, consumers' response to those transfers is predominantly what drives the surge in production and consumption. The exception is when transfers are sent out to those who already have

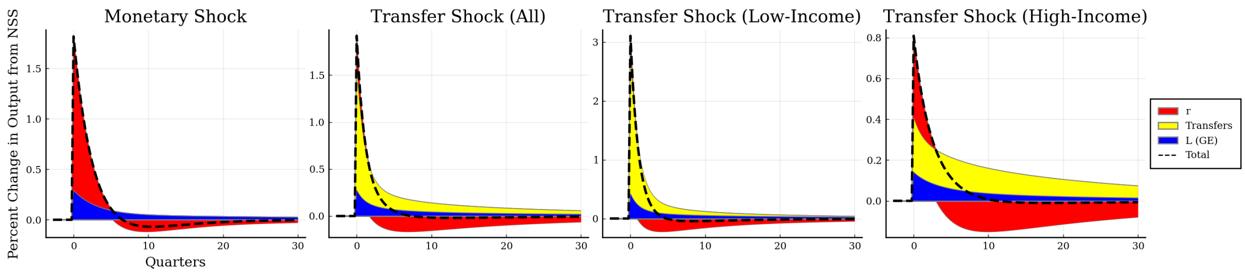


Figure 7: Decompositions of the real output impulse response function in an active fiscal/passive monetary HANK. Each channel represents the heterogeneous agents' response to i) real interest rates and bond prices (in red), ii) transfers (in yellow), and general equilibrium changes in labor demand (in blue). The colored regions add up to the dashed black line.

above-average income. In this last case, the fall in real interest rates due to inflation plays a larger role.[8] At first, the lower return on savings drives households to spend, boosting demand. As inflation erodes the real balances of households below their steady-state level, however, their precautionary savings motives lead them to start saving again, tempering the output boom.

7. Discussion

Because low-income households have low liquid wealth and high marginal propensities to consume, sending deficit-financed transfers to them leads to a sharp boost in output. However, if the central bank does not raise nominal interest rates in response to inflation, then the distribution of transfer recipients has little impact on how much inflation transpires. Inflation accumulates until the nominal assets issued by the government and held by households have returned to steady-state levels, regardless of who received the funds; cumulative inflation is not sensitive to heterogeneity in MPCs.

Transfers to the low-income thus generate larger amounts of GDP relative to the amount of inflation they produced, compared to when the checks go to wealthier high-income segments of the population. This is consistent with the baseline Phillips curve; when output rises quickly, firms take time to adjust their prices and respond to future expected output gaps, not previous ones, leading the overall rise in the price level to trail a sharp rise in

[8]Note that real rates only move due to inflation in these fiscal transfer simulations, as the central bank maintains constant nominal rates.

output. Conversely, this dynamic has strong implications for "sacrifice ratios": abating inflation by cutting transfers to the low-income depresses real GDP by much more than similar inflation abatement accomplished by lump-sum tax increases on the rich.

Monetary policy in an active-fiscal, passive monetary HANK interacts with the price level in roughly the same way as it does in a RANK model with the Cochrane (2018) mechanism. Bondholders as a group are relatively well-insured from idiosyncratic shocks and behave much like the representative agent. Most (85%) of the effect of the change in interest rates on GDP is directly through how the expected path of interest rates changes bond prices and real rates of return. Since many households in the HANK model have few to no assets and have an inelastic demand for assets, they do not propagate this wealth effect, leading monetary policy to have less of an impact on real GDP than in RANK. Once again, heterogeneity in marginal propensities to consume matters for real output, but little for the overall change in the price level following policy changes. As a corollary to this, the ability of the central bank to fight inflation via unexpected changes in its policy rate is still limited when monetary policy is passive; a variant of the "stepping-on-a-rake" dynamic explored by Sims (2011) and Cochrane (2018) still applies.

Studying recent theories of the price level in the context of a heterogeneous agent incomplete markets model illuminates both additional nuance in some aspects and clarifying simplicity in others. On the real output side, the distributional aspects of monetary and fiscal policy are integral for understanding how output responds to each. Unsurprisingly, it matters how a program is targeted, as that targeting affects the timing and magnitude of the change in output. However, the implications of this timing further confound simple intuitions about sacrifice ratios. The intuition that the price level might strongly depend on how some households behave more like "savers" or "spenders" after receiving their checks is also not quantitatively borne out in a HANK model. As Auclert et al. (2018) notes, optimizing agents will eventually want to spend the present value of whatever they receive, such that the present value of iMPCs aggregates to one, even if they smooth that consumption spending over time. Eventually, for the asset market to clear and for the economy to return to its non-stochastic steady-state, inflation occurs to bring nominal private assets back to stable real levels.

When this is the case, one can predict the long-term inflationary impact of a policy without much knowledge of its distributional consequences or implications for employment and output. But is this the case? Less conventional, but perhaps important, theoretical complications could emerge if models contain behavioral agents with MPCs that are truly zero, such as in Auclert et al. (2023b), leading them to act as a permanent real asset sink. Inflation might play a less predictable, and perhaps reduced, role in the equilibrium dynamics of such models. Empirically, there also appears to be an opening for more work examining how inflation does or does not ensue when governments do not have a credible plan to pay down its debt through conventional means following unexpected deficit spending. Ultimately, recent theories of the price level and models with meaningful heterogeneity open up new ways to understand how fiscal and monetary policy interact to influence macroeconomic aggregates – potentially with profound implications for policy in the real world.

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Appendix A. The Phillips Curve and Cumulative Inflation and Output Gaps

Appendix A.1. The Cumulative Inflation and Output and the Phillip's Curve

Defining $\Delta\pi_\infty$ and Δy_∞ as the total cumulative inflation, equation (17) can be integrated forward to write

$$\pi_t = \nu \int_t^\infty e^{-\rho s}\widehat{Y}_s ds$$

Accumulating inflation from time 0 to a terminal time T,

$$\Delta\pi_T^c = \exp\left(\int_0^T \pi_t dt\right) - 1 \approx \int_0^T \pi_t dt = \nu \int_0^T \left(\int_t^\infty e^{-\rho s}\widehat{Y}_s ds\right) dt$$

Note that the order of the integration cannot be interchanged: the timing of the output gaps matter.

Note that if the rate of time discounting ρ is small and the output gaps return to steady-state quickly (as is the case in my models), then the time discounting in the integral is not quantitatively important, such that $\int_0^T e^{-\rho s}\widehat{Y}_s ds \approx \Delta Y_T^c$. Furthermore, the interior integrand can be written as

$$\int_t^\infty e^{-\rho s}\widehat{Y}_s ds = \int_0^\infty e^{-\rho s}\widehat{Y}_s ds - \int_0^t e^{-\rho s}\widehat{Y}_s ds \approx \Delta Y_\infty^c - \Delta Y_t^c$$

such that

$$\Delta\pi_T^c \approx \nu \int_0^T [\Delta Y_\infty^c - \Delta Y_t^c] dt$$

If the gap narrows exponentially, such that $\Delta Y_t^c = (1 - e^{-\alpha t})\Delta Y_\infty^c$, then

$$\Delta\pi_T^c \approx \nu \int_0^T e^{-\zeta t}\Delta Y_\infty^c dt$$

and then taking the limit as $T \to \infty$,

$$\frac{\Delta\pi_\infty^c}{\Delta Y_\infty^c} = \nu \int_0^\infty e^{-\alpha t} dt = \nu \left[-\frac{1}{\zeta}e^{-\alpha t}\right]\Big|_0^\infty = \frac{\nu}{\alpha}$$

In other words, the asymptotic amount of cumulative inflation relative to cumulative output tends to increase with the slope of the Phillips curve, but *decrease* when output rises faster.

More output in a given time increment increases the amount of inflation, but nominal rigidities imply that faster growth in output mean that prices cannot, in a sense keep up. The Phillips curve is forward looking; previous output gaps are already sunk from the perspective of the firm. If a lot of growth happens quickly and then subsides, that past growth no longer matters for period t inflation; all that matters are future output gaps.

Appendix B. Derivations

Appendix B.1. Bond Math

Appendix B.1.1. General Maturities and Formula Derivations

To elaborate more upon the structure of government debt in my model, I more generally assume that the government is able to borrow using long-term nominal bonds of any maturity τ, as in Cochrane (2018). As such, it can pay off existing nominal debt $\tilde{B}$ maturing at time t by either running a primary surplus or by selling new bonds with a maturity of τ at a price of $Q^B_{t,t+\tau}$. The debt flow equation is thus

$$\underbrace{\tilde{B}_{t,t}dt}_{\substack{\text{Debt}\\\text{maturing}\\\text{at time } t}} = \underbrace{p_t(T_t - G_t)dt}_{\text{Surplus}} + \underbrace{\int_0^\infty Q^B_{t,t+\tau} d\tilde{B}_{t,t+\tau} d\tau}_{\substack{\text{Financing from}\\\text{new bond sales}}}$$

I denote the real value of total government debt outstanding at time t as B_t, such that

$$B_t \equiv \frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}$$

I next assume that bonds are purchased and priced not directly by households, but rather by a risk-neutral profit-maximizing investment fund that buys debt from the government and sells shares to the public. The central fiscal theory equation alluded to in the introduction of this paper therefore takes the form presented in Cochrane (2018):

$$\underbrace{\frac{\int_0^\infty Q^B_{t,t+\tau}\tilde{B}_{t,t+\tau}d\tau}{p_t}}_{\substack{\text{Real debt}\\\text{outstanding}}} = \mathbb{E}_t\left[\int_t^\infty e^{-\int_t^\tau r_s ds}[T_\tau - G_\tau]d\tau\right]$$

Each household that holds liquid assets by holding shares in the fund thus effectively owns a cross-sectional slice of the entire government portfolio, and receives whatever interest payments are distributed and absorbs whatever capital gains and losses the government debt accrues.

For the bond portfolio, the total *real* return is the real capital gain on each bond type, weighted by the value of the bonds held, divided by the real value of the entire portfolio:

$$dR_t = \frac{\int_0^\infty \left[d\left(\frac{Q_{t,t+\tau}}{p_t}\right) / \frac{Q_{t+\tau}}{p_t} \right] \frac{Q_{t,t+\tau}}{p_t} \tilde{B}_{t,t+\tau} d\tau}{B_t}$$

$$\Rightarrow \quad B_t dR_t = \int_0^\infty d\left(\frac{Q_{t,t+\tau}}{p_t}\right) \tilde{B}_{t,t+\tau} d\tau$$

Such that

$$dB_t = d\left[\frac{\int_0^\infty Q_{t,t+\tau} \tilde{B}_{t,t+\tau} d\tau}{p_t}\right] = \underbrace{\frac{\int_0^\infty Q_{t,t+\tau} d\tilde{B}_{t,t+\tau} d\tau}{p_t} - \frac{\tilde{B}_t}{p_t} dt}_{-(T_t - G_t)dt} + \underbrace{\int_0^\infty \tilde{B}_{t,t+\tau} d\left(\frac{Q_{t,t+\tau}}{p_t}\right) d\tau}_{B_t dR_t}$$

It thus follows that

$$dB_t = -(T_t - G_t)dt + B_t dR_t$$

The first term is the primary deficit, while the second is the ex-post real rate of return on the bond portfolio. This ex-ante return will then be the expected return on the nominally riskless bonds, plus whatever capital gain has been unexpectedly accrued over the time increment.

Again as in Cochrane (2018), I make the simplifying assumption that the government issues and rolls over debt such that the density of of government liabilities by maturity is always exponentially distributed with a rate of ω, such that the cumulative distribution of outstanding government treasury maturities τ is $CDF(\tau) = 1 - e^{-\omega\tau}$ and the density function is $PDF(\tau) = \omega e^{-\omega\tau}$. Additionally, I make the simplifying assumption that in the non-stochastic steady-state of the model, all households effectively hold the same representative slice of government debt by owning shares of a competitive profit-maximizing mutual fund, just in varying amounts. For an individual holding a unitary share of the total government portfolio, their assets entitle them to a payment of ωdt almost immediately (this is the shortest-term debt being repaid), plus payments of $\omega e^{-\omega\tau} dt$ for all periods thereafter. The entire bond portfolio is then effectively a perpetuity which pays out a geometrically declining coupon $\omega e^{-\omega\tau} dt$ at each time $t + \tau$ for the rest of time.

The nominal bond price of the entire portfolio will then be

$$q_t^B = \int_0^\infty e^{-\tau y_t} \omega e^{-\omega\tau} d\tau = \int_0^\infty \omega e^{-\tau(\omega+y_t)} d\tau = -\frac{\omega}{\omega+y_t} e^{-u}|_0^\infty = \frac{\omega}{\omega+y_t}$$

The nominal rate of return on the bond will be the the dividend yield, plus the capital gain.

$$dR_t^{nom} = \frac{(\omega - \omega q_t^B)dt + dq_t^B}{q_t^B} = y_t dt + \frac{dq_t^B}{q_t^B}$$

It then follows that if the ex-ante nominal rate of return is dR_t^{nom} is $i_t dt$ in expectation

$$i_t dt = \mathbb{E}_t[dR_t^{nom}] = y_t dt + \frac{\mathbb{E}_t[dq_t^B]}{q_t^B}$$

I define $\delta_{qB,t} = dq_t^B - \mathbb{E}_t[dq_t^B]$ as the unexpected gain in bond prices, which must in turn be equal to the ex-post nominal rate of return minus the expected (ex-ante) one:

$$\frac{\delta_{qB,t}}{q_t^B} \equiv dR_t^{nom} - i_t dt = \frac{dq_t^B - \mathbb{E}_t[dq_t^B]}{q_t^B}$$

Since the nominal rate will be the real one, plus inflation:

$$dR_t^{nom} = dR_t + \pi_t dt$$

$$\Rightarrow \quad \frac{\delta_{qB,t}}{q_t^B} - \pi_t dt = dR_t - i_t dt$$

$$\Rightarrow \quad dR_t = \frac{\delta_{qB,t}}{q_t^B} + (i_t - \pi_t)dt$$

The valuation equation becomes

$$dB_t = -(T_t - G_t)dt + B_t\left[i_t - \pi_t\right]dt + \frac{\delta_{qB,t}}{q_t^B} B_t \tag{B.1}$$

To derive the equation governing nominal bond prices, it also follows that if

$$dR_t^n = \frac{\omega dt + dq_t^B}{q_t^B} - \omega dt$$

such that

$$q_t^B dR_t^n = \omega dt + dq_t^B - \omega q_t^B dt$$

then in expectation

$$E_t[dq_t^B] = q_t^B \left(\mathbb{E}_t[dR_t^n] + \omega dt - \frac{dt}{q_t^B} \right)$$
$$\Rightarrow \quad E_t[dq_t^B] = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt \tag{B.2}$$

and so bond prices evolve according to

$$dq_t^B = q_t^B \left(i_t + \omega - \frac{\omega}{q_t^B} \right) dt + \delta_{qB,t}$$

Appendix B.2. Wage Phillips Curve

This is a continuous-time version of Auclert et al. (2018), *The Intertemporal Keynesian Cross*. Say a labor-aggregator hires labor from households to create an aggregate unit of input labor:

$$L_{k,t} = \int_0^1 (z_i h_{ikt}) di$$

And labor from each union is differentiated with elasticity of substitution ε_ℓ:

$$L_t = \left(\int_0^1 L_{k,t}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}}$$

Let W_t be the nominal wage paid by employers to labor-aggregators, and let the labor-aggregator pay its workers a nominal wage of $W_{k,t}$. Labor-aggregating firms thus hire according to

$$\max_{\{L_{k,t}\}_{k \in [0,1]}} W_t \left(\int_0^1 L_{k,t}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1}} - \int_0^1 W_{k,t} L_{k,t} dk$$

such that from the FOCs, the demand for labor from union k is

$$W_t \left(\int_0^1 L_{k,t}^{\frac{\varepsilon_\ell - 1}{\varepsilon_\ell}} dk \right)^{\frac{\varepsilon_\ell}{\varepsilon_\ell - 1} - 1} L_{k,t}^{-\frac{1}{\varepsilon_\ell}} - W_{k,t} = 0$$

$$W_t L_t^{\frac{1}{\varepsilon_\ell}} L_{k,t}^{-\frac{1}{\varepsilon_\ell}} = W_{k,t}$$

$$W_t L_t^{\frac{1}{\varepsilon_\ell}} = W_{k,t} L_{k,t}^{\frac{1}{\varepsilon_\ell}}$$

$$\Rightarrow \quad \frac{L_{k,t}}{L_t} = \left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell}$$

Unions face nominal wage adjustment costs:

$$\frac{\theta_w}{2}\int_0^1 \pi_{w,k}^2 dk, \quad \text{where} \quad \pi_{w,k} = \frac{dW_{k,t}}{dt}\frac{1}{W_{k,t}}$$

The labor union k sets wages to maximize its members' lifetime utilities:

$$J_t^w(W_{k,t}) = \max_{\pi_{k,t}^w} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[\int\int\left\{\frac{c(a,z)^{1-\gamma}}{1-\gamma} - \frac{h(a,z)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z)da\ dz - \frac{\theta_w}{2}(\pi_{k,t}^w)^2\right]dt$$

$$\text{s.t.} \ \frac{dW_t}{dt} = \pi_t^w W_t$$

$$L_{k,t} = \int_0^1 z_i h_{ikt} di$$

$$\frac{L_{k,t}}{L_t} = \left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell}$$

Where the third equation follows from the first-order conditions from the households.

The HJB is then (suppressing the value function's arguments for brevity)

$$\rho J_t^w = \left[\int\int\left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{k,t})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z)da\ dz - \frac{\theta_w}{2}(\pi_{k,t}^w)^2\right] + \frac{\partial J_t^w}{\partial W_{k,t}}\pi_t^w W_{k,t} + \frac{\partial J_t^w}{\partial t}$$

The FOC for wage inflation is then

$$-\theta_w \pi_{k,t}^w + \frac{\partial J^w(W_{k,t})}{\partial W_{k,t}} W_{k,t} = 0$$

$$\Rightarrow \quad \frac{\partial J^w(W_{k,t})}{\partial W_{k,t}} = \theta_w \frac{\pi_{k,t}^w}{W_{k,t}}$$

Taking the total differential of the marginal value of wages,

$$d\left(\frac{\partial J_t^w(W_{k,t})}{\partial W_{k,t}}\right) = \partial_{W_{k,t}}^2 J_t^w dW_{k,t} + \partial_{W_{k,t}}\partial_t J_t^w dt$$

and doing the same to the LHS of the wage inflation FOC,

$$d\left(\theta_w \frac{\pi_t^w}{W_{k,t}}\right) = \frac{\theta_w}{W_{k,t}} d\pi_t^w - \frac{\theta_w \pi_t^w}{W_{k,t}^2} dW_{k,t}$$

I can equate the two:

$$\frac{\theta_w}{W_{k,t}} d\pi_t^w - \frac{\theta_w \pi_t^w}{W_{k,t}^2} dW_{k,t} = \partial_{W_{k,t}}^2 J^w dW_{k,t} + \partial_t \partial_{W_{k,t}} J_t^w dt.$$

Taking expectations and dividing by dt yields

$$\frac{\theta_w}{W_{k,t}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{k,t}} \underbrace{\frac{dW_{k,t}}{dt} \frac{1}{W_{k,t}}}_{\pi_{k,t}^w} = \partial_{W_{k,t}}^2 J_t^w \frac{dW_{k,t}}{dt} + \partial_{W_{k,t}} \partial_t J_t^w$$

such that

$$\frac{\theta_w}{W_{k,t}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{k,t}} \pi_t^w = \partial_{W_{k,t}}^2 J^w \pi_t^w W_{tk} + \partial_{W_{k,t}} \partial_t J_t^w \tag{B.3}$$

Next, the Envelope condition stipulates that

$$\rho \partial_{W_{k,t}} J_t^w = \left[\int\int \partial_{W_{k,t}} \left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{k,t})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\} \mu_t(a,z) da\ dz\right]$$
$$+ \partial_{W_{k,t}}^2 J^w \pi_t^w W_{k,t} + \partial_{W_{k,t}} J^w (W_{k,t}) \pi_t^w + \partial_{W_{k,t}} \partial_t J_t^w$$

Substituting in (B.3),

$$\rho \partial_{W_{k,t}} J_t^w = \left[\int\int \partial_{W_{k,t}} \left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{k,t})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\} \mu_t(a,z) da\ dz\right]$$
$$+ \partial_{W_{k,t}} J_t^w (W_{k,t}) \pi_t^w + \frac{\theta_w}{W_{k,t}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{k,t}} \pi_t^w$$

and then the FOC,

$$\rho \theta_w \frac{\pi_{k,t}^w}{W_{k,t}} = \left[\int\int \partial_{W_{k,t}} \left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{k,t})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\} \mu_t(a,z) da\ dz\right]$$
$$+ \theta_w \frac{\pi_{k,t}^w}{W_{k,t}} \pi_t^w + \frac{\theta_w}{W_{k,t}} \frac{\mathbb{E}_t[d\pi_t^w]}{dt} - \frac{\theta_w \pi_t^w}{W_{k,t}} \pi_t^w$$

it follows that

$$\rho\pi^w_{k,t} = \frac{W_{k,t}}{\theta_w}\left[\int\int \partial_{W_{k,t}}\left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma} - \frac{h(a,z;W_{k,t})^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right\}\mu_t(a,z)da\ dz\right] + \frac{\mathbb{E}_t[d\pi^w_t]}{dt}. \quad \text{(B.4)}$$

From the households' envelope condition, the change in utility from wages will be equal to the marginal utility, times the change in earnings:

$$\partial_{W_{k,t}}\left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma}\right\} = c(a,z)^{-\gamma}(1-\tau)\partial_{W_{k,t}}\left(z\frac{W_{k,t}}{P_t}h(a,z)\right)$$

Where if households uniformly supply their labor to union k, and unions internalize their labor's demand:

$$h_{ikt}(a,z) = \frac{1}{Z}L_{k,t} = \frac{1}{Z}\left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell} L_t$$

$$\begin{aligned}
\Rightarrow \quad \partial_{W_{k,t}}\left\{\frac{c(a,z;W_{k,t})^{1-\gamma}}{1-\gamma}\right\} &= c(a,z)^{-\gamma}(1-\tau)\partial_{W_{k,t}}\left(z\frac{W_{k,t}}{P_t}\left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell}\frac{1}{Z}L_t\right) \\
&= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{Z}\frac{1}{W_{k,t}}\left(\frac{W_{k,t}}{P_t}\left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell} L_t\right) \\
&= c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{Z}\frac{1}{P_t}L_{k,t}
\end{aligned}$$

For the effect of wages on labor disutility, I can directly evaluate

$$\partial_{W_{k,t}}h(a,z) = \frac{1}{Z}\partial_{W_{k,t}}\left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell} L_t = -\varepsilon_\ell\frac{1}{Z}\frac{1}{W_{k,t}}\left(\frac{W_t}{W_{k,t}}\right)^{\varepsilon_\ell} L_t = -\varepsilon_\ell\frac{1}{Z}\frac{L_{k,t}}{W_{k,t}}$$

Plugging in the results into (B.4),

$$\rho\pi^w_{k,t} = \frac{W_{k,t}}{\theta_w}\left[\int\int\left\{c(a,z)^{-\gamma}(1-\tau)(1-\varepsilon_\ell)\frac{z}{Z}\frac{1}{P_t}L_{k,t} + \frac{1}{Z}h(a,z)^{\frac{1}{\eta}}\varepsilon_\ell\frac{L_{k,t}}{W_{k,t}}\right\}\mu_t(a,z)da\ dz\right] + \frac{\mathbb{E}_t[d\pi^w_t]}{dt}$$

$$\rho\pi^w_{k,t} = \frac{\varepsilon_\ell}{\theta_w}\frac{L_{k,t}}{Z}\int\int\left\{h(a,z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)z\frac{W_{k,t}}{P_t}c(a,z)^{-\gamma}\right\}\mu_t(a,z)da\ dz + \frac{\mathbb{E}_t[d\pi^w_t]}{dt}$$

Leading to the wage Phillips Curve

$$\frac{\mathbb{E}_t[d\pi^w_t]}{dt} = \rho\pi^w_t - \frac{\varepsilon_\ell}{\theta_w}\frac{L_t}{Z}\int\int\left(h(a,z)^{\frac{1}{\eta}} - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t c(a,z)^{-\gamma}\right)\mu_t(a,z)\ da\ dz \quad \text{(B.5)}$$

where $w_t \equiv \frac{W_{k,t}}{P_t}$ is the real wage in the symmetric equilibrium where $W_{k,t} = W_t \; \forall k \in [0,1]$.

Appendix C. Solving Bayer and Luetticke (2020) in Continuous Time

This section is best read after having already read Achdou et al. (2021), Ahn et al. (2018), and particularly Bayer and Luetticke (2020) as background; the below section largely amounts to a brief sketch of adapting Bayer and Luetticke (2020) to continuous time. For notational brevity, I write the infinitessimal generator operator of the concentrated Hamilton Jacobi Bellman equation as

$$\begin{aligned}\mathcal{D}[V] =& \lim_{t\downarrow 0} \frac{\mathbb{E}_t^{a,z}[V_t(a_{t+dt}, z_{t+dt})] - V_t(a_t, z_t)}{dt} \\ =& \frac{\partial V_t}{\partial a}(a, z; \mu, \zeta)\Big[(1-\tau)w_t z h_t(a, z) + T_t(a, z) - c_t(a, z) + r_t(a)a\Big] \\ &+ \frac{\partial V_t}{\partial z}(a, z; \mu, \zeta) z \left[\frac{1}{2}\sigma_z^2 - \theta_z \log(z)\right]\end{aligned}$$

where the expectation operator is taken with respect to only the idiosyncratic variables. As in Achdou et al. (2021), I write the adjoint operator (which describes the Kolmogorov forward equation of the idiosyncratic state distribution) as $\mathcal{D}^*$, where the KFE operator is the adjoint of the maximized HJB operator in L^2 space. Additionally, I write expectation errors for a jump variable "J" as $d\delta_{J,t}$, such that $d\delta_{J,t} = dJ_t - \mathbb{E}_t[dJ_t]$.

Suppose aggregate shocks in the economy evolve according to

$$d\zeta_t = -\Theta_\zeta \zeta_t dt + d\epsilon_{\zeta,t}. \tag{C.1}$$

A sequential equilibrium following a perturbation from the steady-state $W_{\zeta,0}$ is a resulting path of aggregate shocks $\{\zeta_t\}_{t\geq 0}$, a series of value functions $\{V_t(a, z)\}_{t\geq 0}$, consumption decisions and labor allocations $\{c_t(a, z), h_t(a, z)\}_{t\geq 0}$, distributions $\{\mu_t(a, z)\}_{t\geq 0}$, outstanding government debt $\{B_t\}_{t\geq 0}$, wages $\{w_t\}_{t\geq 0}$, nominal and real interest rates $\{i_t, r_t\}_{t\geq 0}$, bond prices $\{q_t^B\}_{t\geq 0}$, and inflation rates $\{\pi_t\}_{t\geq 0}$ where

$$dV_t(a, z) = \left\{\rho V_t(a, z) - \Big[u(c_t(a, z)) - v(h_t(a, z)) + \mathcal{D}[V]\Big]\right\}dt - \frac{\partial V_t(a, z)}{\partial a} d\delta_{qB,t} + d\delta_{V(a,z),t} \tag{C.2}$$

and if $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, it follows that the FOC for consumption is

$$c_t(a,z)^{-\gamma} = \frac{\partial V_t}{\partial a}(a,z). \tag{C.3}$$

The distribution evolves according to

$$d\mu_t(a,z) = \mathcal{D}^*[\mu]dt - \frac{\partial}{\partial a}\big(\mu_t(a,z)d\delta_{qB,t}\big) \tag{C.4}$$

while labor is supplied to meet market demand:

$$\begin{aligned} \frac{1}{Z}h_{NSS}(a,z)^\eta &= \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zwc_{NSS}(a,z)^{-\gamma} \\ h_t(a,z) = h_{NSS}(a,z)&\frac{d\mu_{NSS}}{d\mu_t}(a,z) + \frac{1}{Z}(L_t - L_{NSS}) \end{aligned} \tag{C.5}$$

Goods inflation must be consistent with the goods market Phillips Curve derived from the firms' profit maximization problem:

$$d\pi_t = \left(r_t\pi_t - \frac{\varepsilon}{\theta_\pi}[m_t - m^*]\right)dt + d\delta_{\pi,t} \tag{C.6}$$

while wage inflation is dictated by the labor market Phillips Curve

$$d\pi_t^w = \left\{\rho\pi_t^w - \frac{\varepsilon_\ell}{\theta_w}L_t \int\int \left(v'(h(a,z)) - \frac{\varepsilon_\ell - 1}{\varepsilon_\ell}(1-\tau)zw_t u'(c(a,z))\right) da\ dz\right\} dt + d\delta_{\pi^w,t} \tag{C.7}$$

Real wages then follow

$$dw_t = (\pi_t^w - \pi_t)w_t dt \tag{C.8}$$

The government's budget constraint must satisfy

$$dB_t = -(T_t - G_t)dt + r_t B_t dt + \frac{d\delta_{qB,t}}{q_t^B}B_t \tag{C.9}$$

where nominal bond prices and equity prices satisfy

$$dq_t^B = q_t^B\left(i_t + \omega - \frac{\omega}{qB,t}\right)dt + d\delta_{qB,t} \tag{C.10}$$

Equilibrium must also be consistent with the Fisher equation, the marginal cost equation, and the profit equation:

$$r_t = i_t - \pi_t \tag{C.11}$$

$$m_t = w_t \tag{C.12}$$

$$\Pi_t = [1 - m_t]Y_t \tag{C.13}$$

All goods consumed must be produced:

$$Y_t = L_t \tag{C.14}$$

and the idiosyncratic variables must aggregate:

$$C_t = \int_0^\infty \int_{\underline{a}}^\infty c_t(a, z)\mu_t(a, z)da\ dz \tag{C.15}$$

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty zh_t(a, z)\mu_t(a, z)da\ dz \tag{C.16}$$

Finally, goods and financial markets must clear:

$$Y_t = C_t + \int_0^\infty \int_{\underline{a}}^\infty -\mathbf{1}_{\{a<0\}}\Delta_r a\mu_t(a, z)da \tag{C.17}$$

$$B_t = \int_0^\infty \int_{\underline{a}}^\infty a\mu_t(a, z)da\ dz \tag{C.18}$$

It's then possible to write a vector of variables as

$$X_{C,t} = (V_t(a, z),\ \pi_t,\ \pi_t^w,\ q_t^B)',$$

the set of state variables as

$$X_{S,t} = (\mu_t(a, z),\ B_t,\ w_t, \zeta_t)',$$

and the vector of static constraints as

$$X_{L,t} = (Y_t,\ L_t)',$$

(where many of the static constraints like the Fisher equation and the employment rules can be re-written to solve out the other static variables from the model). Stacking the controls, states, and static variables, I write

$$X_t = (X_{C,t}, X_{S,t}, X_{L,t})'$$

where dX_t represents the differentials of X_t. Using this succinct notation, the entire system (C.1-C.18) can be written as

$$\Gamma_0 dX_t = \Omega(X_t, d\delta_{X,t}, d\epsilon_{\zeta,t}) \tag{C.19}$$

where the rows of Γ_0 corresponding to static constraints are equal to zero.

I discretize the partial differential equations on the computer in the non-stochastic steady-state where $X_t = X_{NSS}$, $dX_t = 0$, $d\delta_{X,t} = 0$, and $d\epsilon_{\zeta,t} = 0$, using the finite-differences methodology described in Achdou et al. (2021). This entails discretizing (C.19) via an upwind finite difference approximation for the partial derivatives along an asset grid (which I index by $i \in I \equiv \{1, \ldots, N_a\}$) and an income grid (which I index by $j \in J \equiv \{1 \ldots, N_z\}$). The tensor $V_{i,j,nss}$ then approximates the value function $V_{NSS}(a_i, z_j)$ in the discretized state space, while the tensor $\mu_{i,j,nss}$ approximates the distribution $\mu_{NSS}(a_i, z_j)$.

Before proceeding, I find it useful to define $\widehat{X}_t \equiv X_t - X_{NSS}$ as either the level deviations or the log deviations of the variables from their values in the non-stochastic steady-state. As such, the complete system can be rewritten to become

$$\Gamma_0 d\widehat{X}_t = \widehat{\Omega}(\widehat{X}_t, d\delta_{X,t}, d\epsilon_{\zeta,t}) \tag{C.20}$$

where the arguments are the deviation terms. The steady-state thus satisfies $\widehat{\Omega}(\mathbf{0}) = \mathbf{0}$. I then proceed to solve for the dynamics of the economy following aggregate shocks. Practically, the dimensionality of the discretized value functions and distributions necessitate dimension reduction. However, for clarity, I first describe the process *without* dimension reduction.

Appendix C.1. Without Dimension Reduction

With the non-stochastic steady-state (NSS) in hand, I then calculate the numerical Jacobian of the system at the NSS using automatic differentiation. Differentiating the entire system with respect to just the arguments in X_t alone, I can write the Jacobian of the system with respect to its X_t variables at the non-stochastic steady-state as

$$\Gamma_{X,X} \equiv \nabla_X \widehat{\Omega}(\mathbf{0})$$

While the derivatives of the system with respect to the expectation errors and the perturbations are

$$\Gamma_{X,\delta} \equiv \nabla_{d\delta} \Omega(\mathbf{0})$$

$$\Gamma_{X,W} \equiv \nabla_{dW_\zeta} \Omega(\mathbf{0})$$

A first-order Taylor expansion of the system around the steady-state without any shocks (and where $d\widehat{X}_t = 0$) is then

$$\Gamma_0 d\widehat{X}_t = \Gamma_{X,X} \widehat{X}_t dt + \Gamma_{X,\delta} d\delta_{X,t} + \Gamma_{X,W} d\epsilon_{\zeta,t} + \mathcal{O}(\|\widehat{X}_t, d\delta_{X,t}, d\epsilon_{\zeta,t}\|^2)$$

I then solve

$$\Gamma_0 d\widehat{X}_t = \Gamma_{X,X} \widehat{X}_t dt + \Gamma_{X,\delta} d\delta_{X,t} + \Gamma_{X,W} d\epsilon_{\zeta,t} \tag{C.21}$$

using the generalized eigenvalue methodology described in Sims (2002). If the system has more stable generalized eigenvalues than it has control variables, the dimensionality of the linear subspace being used to approximate the system's stable manifold is too large to ensure that the dynamics are unique, such that multiple equilibria are possible (sunspots). If the system has fewer stable eigenvalues than state variables, then the equilibrium cannot exist. I verify that the number of stable eigenvalues in my system matches the number of state variables, such that the solution exists and is unique.

While straightforward, this approach is too computationally costly to be feasible with the number of gridpoints that I employ to solve my full model. As such, I use the dimension reduction strategy of Bayer and Luetticke (2020) before calculating the Jacobian of (C.20).

Appendix C.2. With Dimension Reduction

I write the 2-dimensional discrete cosine transform (DCT) of a 2-dimensional array A as $\theta^A = \text{DCT}(A)$, where its inverse $\text{DCT}^{-1}(\theta^A) = A$. I can write the transformation of the value function in the non-stochastic steady-state as

$$\{\theta^V_{(i,j),nss}\}_{(i,j)\in I\times J} = \text{DCT}(\{V_{(i,j),nss}\}_{(i,j)\in I\times J})$$

I then compute the "energy" (to use the terminology of Bayer and Luetticke (2020)) of the $\theta^V_{i,j,nss}$ coefficients as

$$E_{ij} = \frac{[\theta^V_{(i,j),nss}]^2}{\sum_{(i,j)\in I\times J}[\theta^V_{(i,j),nss}]^2}$$

Sorting the coefficients by their energy from greatest to least, I then identify those coefficients that contain a cumulative $1-\kappa$ share of the coefficients' energy, where κ is a small number. I label the set of these coefficients (which are effectively the ones with the largest absolute value) as Θ_E; these coefficients explain most of the variation of the value function in the steady-state.

As in Bayer and Luetticke (2020), I then move toward constructing a perturbation solution of the equilibrium system, but perturbing only high-energy coefficients in Θ_E. Otherwise, I keep the lower-energy coefficients constant, at their steady-state values:

$$\tilde{\theta}^V_{i,j,t} = \theta^V_{(i,j),t} + \mathbf{1}_{\{(i,j)\in\Theta_E\}}\widehat{\theta}^V_{(i,j),t}$$

where $\widehat{\theta}^V_{i,j,t}$ is the coefficient's deviation at time t from its NSS value.

The DCT is a linear operator. As such, I can write the differentials of the coefficients as

$$\{d\theta^V_{(i,j),t}\}_{(i,j)\in I\times J} = d\left[\text{DCT}(\{V_{(i,j),t}\}_{(i,j)\in I\times J})\right] = \{d\theta^V_{(i,j),nss}\}_{(i,j)\in I\times J} = \left[\text{DCT}(\{dV_{(i,j),nss}\}_{(i,j)\in I\times J})\right]$$

and similarly I write

$$d\tilde{\theta}^V_{(i,j),t} = \mathbf{1}_{\{(i,j)\in\Theta_E\}} d\theta^V_{(i,j),t}$$

By perturbing only the $|\Theta_E|$ largest-magnitude coefficients instead of the full $N_a \times N_z$ elements of the discretized value function, I can greatly reduce the dimensionality of the problem. Of course, this only reduces the number of control variables. To reduce the number of state variables in the distribution, I also employ the fixed copula transformation of Bayer and Luetticke (2020).

I write the discretized joint cumulative distribution function $F_{\mu(a_i,z_j)}$, and the marginal CDFs as $F_{\mu(a_i)}$ and $F_{\mu(z_j)}$. The copula is then the joint distribution interpolated onto the marginal ones:

$$\text{Cop} = \text{Interp}(\{F_{\mu(a_i,z_j),nss}\}_{ij}, \{F_{\mu(a_i),nss}\}_i, F_{\mu(z_j),nss}\}_j)$$

where the nss subscript denotes the steady-state values. It then follows that $\text{Cop} : [0,1] \times [0,1] \to [0,1]$ maps cumulative marginal distributions to a joint distribution, as predicted by the rank correlations of the steady-state. Outside of the steady-state, I then approximate the joint cumulative distribution $F_{\mu(a_i,z_j),t}$ at time t as

$$F_{\mu(a_i,z_j),t} \approx \text{Cop}(F_{\mu(a_i),t}, F_{\mu(z_j),t}),$$

from which the marginal joint *density* function μ_{ij} may be derived. Using this object, I can then iterate the Kolmogorov Forward Equation to obtain $d\mu_{ij}$, which can be integrated (or summed, since the functions are discretized) to obtain the evolution of the differentials

$$\{(dF_{\mu(a_i),t}, dF_{\mu(z_j),t})\}_{ij}.$$

As Bayer and Luetticke (2020) note, this approximation allows me to track only the N_a and N_z dimensional marginal CDFs instead of their joint one to describe the economy, so long as the rank correlations outside of the steady-state are similar to those represented in the steady-state (which Bayer and Luetticke (2020) show is generally the case in Bewley-Aiyagari models).

I then define the dimension-reduced set of controls as

$$\tilde{X}_{C,t} = (\{\tilde{\theta}^V_{i,j,t}\}_{(i,j)\in\Theta_E},\ \pi_t,\ \pi^w_t,\ q^B_t)'$$

and the dimension-reduced set of states as

$$\tilde{X}_{S,t} = (\{F_{\mu(a_i),t}\}_i, \{F_{\mu(z_j),t}\}_j,\ B_t,\ w_t, \zeta_t)',$$

Once again stacking the reduced controls, states, and static variables, I write

$$\tilde{X}_t = (\tilde{X}_{C,t}, \tilde{X}_{S,t}, X_{L,t})'$$

and the system (C.19) is approximated by a smaller one:

$$\tilde{\Gamma}_0 d\tilde{X}_t = \tilde{\Omega}(\tilde{X}_t, d\delta_{X,t}, d\epsilon_{\zeta,t})$$

where $\tilde{\Omega}$ calculates the value function and joint distribution given the DCT coefficients and the marginal distribution, feeds them back into the original Ω function, and then from there recovers the resulting truncated DCT coefficients and marginal CDFs' time differentials. Just like before, this system can also be written in terms of just the differences (or log differences) of the variables from their non-stochastic steady-state values. The rest of the linearization steps and solution methods then proceed exactly in the same manner as they do in the version without dimension reduction, as reviewed in the prior subsection of this appendix.

Appendix C.3. Numerical Accuracy

I solve the model over a unifrom grid of $N_a = 50$ points from -1 to 60 and $N_z = 40$ grid points from 0.01 to 5.5.

The aggregate law of motion (5) can be used to track the evolution of the market value of government debt, but since households hold the government's bonds as assets, the private sector's total bond position may be calculated by using the Kolmogorov forward equations (19) and aggregating using (21). To assess the accuracy of the model, I calculate the evolution of the stock of government debt both ways, and then observe the percentage difference as a test of my model's numerical accuracy. Overall, the errors in the simulated time series are on the order of 5×10^{-6} in the fiscal policy experiments, where the nominal price of government bonds does not jump.

In the monetary policy experiments, however, the use of a first-order Taylor expansion around where $d\delta_{q,t} = 0$ for the value function and the KFE equation introduces additional numerical errors. When interest rates fall by 1%, the increase in the value of the government's debt should match the jump in bond prices on impact, as neither the price level nor the number of bonds outstanding jump on impact. However, while the bond price jumps 4.45%, the value of those real bonds B_0 aggregated from the KFE equation jumps by only 4.23%, a roughly 5% (0.22 percentage points) difference between the two responses. To reduce this error, I re-scale the debt impulse response function after linearization so that the initial bond and price movements match in the moment that the shock is realized.